Axler 3.C

Objectives

1. Define the matrix of a linear transformation

2. Relate this to graph theory

Recall A linear transformation $T: V \longrightarrow W$ is completely determined by its behavior on a basis $\{v_1, \ldots, v_n\}$ of V.

Pf. Any $v \in V$ can be written $v = a_1 v_1 + \ldots + a_n v_n$ for some $a_i \in F$.

$$\begin{aligned} T(v) &= T(a_1 v_1 + \ldots + a_n v_n) \\ &= T(a_1 v_1) + \ldots + T(a_n v_n) \\ &= a_1 T(v_1) + \ldots + a_n T(v_n). \end{aligned}$$

$\square$

If we choose a basis $\{w_1, \dots, w_m\}$ of W, we can find scalars $a_{11}, a_{21}, \dots, a_{m1} \in F$

$$\begin{array}{ccccc} T(v_1) & T(v_2) & \cdots & T(v_n) \\ \| & \| & & \| \\ a_{11}w_1 & a_{12}w_1 & & a_{1n}w_1 \\ + & + & \ddots & + \\ \vdots & \vdots & & \vdots \\ + & + & & + \\ a_{m1}w_m & a_{m2}w_2 & \cdots & a_{mn}w_n \end{array}$$

We organize the coefficients $\{a_{ij}\}$ in a <u>matrix</u>

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & & v_n \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

$n \times m$ matrix

$\left.\right\}$ The matrix of the linear transformation T w.r.t. the basis $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_m\}$.

Example $T: \mathbb{R}^3 \to \mathbb{R}^2$

$\{v_1, v_2, v_3\}$ $\{w_1, w_2\}$

$$\left.\begin{aligned} T(v_1) &= w_1 \\ T(v_2) &= 2w_1 - w_2 \\ T(v_3) &= 3w_2 + 0w_1 \end{aligned}\right\} \text{has matrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

$$T(\underbrace{av_1 + bv_2 + cv_3}) = aT(v_1) + bT(v_2) + cT(v_3)$$

$$= aw_1 + 2bw_1 - bw_2 + 3cw_2$$

$$= (\underbrace{a+2b})w_1 + (\underbrace{-b+3c})w_2$$

w/matrices: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b + 0c \\ 0 \cdot a - 1 \cdot b + 3c \end{bmatrix} = \begin{bmatrix} a+2b \\ -b+3c \end{bmatrix}$

RECALL The composition of $T: V \to W$ and $S: X \to V$ is $(T \circ S)(x) = T(S(x))$.

What does this look like with matrices??

Choose a basis $\{x_1, \ldots, x_\ell\}$ for X.

$T(v_i) = a_{1i} w_1 + \ldots + a_{mi} w_m$

Define $S(x_i) = b_{1i} v_1 + \ldots + b_{ni} v_i$.

Then $(T \circ S)(x_i) = T(S(x_i))$ by definition

$$= T(b_{1i} v_1 + \ldots + b_{ni} v_i)$$

$$= T\left(\sum_{j=1}^{n} b_{ji} v_j\right)$$

$$= \sum_{j=1}^{n} b_{ji} T(v_j)$$

$$= \sum_{j=1}^{n} b_{ji} \left(\sum_{k=1}^{m} a_{kj} w_k \right) \quad \text{(distributivity)}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} b_{ji} a_{kj} w_k$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{n} b_{ji} a_{kj} w_k \quad \text{(commutativity check this!)}$$

$$= \sum_{k=1}^{m} w_k \left(\sum_{j=1}^{n} b_{ji} a_{kj} \right)$$

$T \circ S$ is the linear transformation defined by

$$(T \circ S)(x_i) = \sum_{k=1}^{m} \left(\underbrace{\sum_{j=1}^{n} a_{kj} b_{ji}} \right) w_k$$

What does this look like on matrices?

The matrix of $T \circ S : X \to W$

$$\begin{array}{c c} & \begin{array}{ccccc} x_1 & x_2 & \cdots & x_i & x_\ell \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_k \\ \vdots \\ w_m \end{array} & \left[\begin{array}{ccccc} & & & \vdots & \\ & & & \vdots & \\ & & & \vdots & \\ \cdots & \cdots & \cdots & \bigcirc & \\ & & & & \end{array} \right] \end{array}$$

$$\underbrace{\sum_{j=1}^{n} a_{kj} b_{ji}}$$

$$\underbrace{\begin{array}{cc} & \\ k \rightarrow & \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \end{array}}_{T} \underbrace{\begin{array}{c} \downarrow i \\ \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} \end{array}}_{S} = \underbrace{\begin{array}{cc} \downarrow i & \\ \begin{bmatrix} & & \\ & \circ & \\ & & \end{bmatrix} & \leftarrow k \end{array}}_{T \circ S}$$

$$\sum_{j=1}^{n} a_{kj} b_{ji} = \underbrace{a_{k1} b_{1i}} + \underbrace{a_{k2} b_{2i}} + \ldots + \underbrace{a_{kn} b_{ni}}$$

This is the "usual" matrix multiplication!

<u>CHECK</u> Matrix multiplication is <u>not</u> commutative.

<u>CHECK</u> If $\mathcal{M}(S)$ the matrix of S and $\mathcal{M}(T)$

the matrix of T, $\mathcal{M}(S) + \mathcal{M}(T) = \mathcal{M}(S+T)$

and $a \in \mathbb{F}$, $\mathcal{M}(aS) = a \cdot \mathcal{M}(S)$.

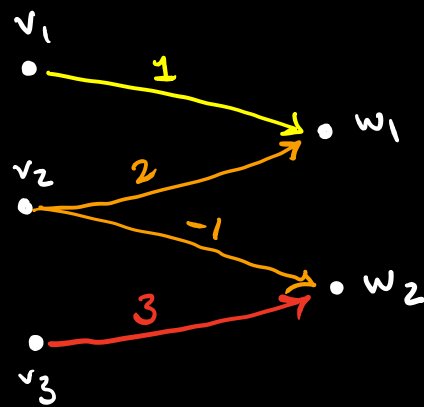
How else can we visualize a linear transformation?

PATHS (graphs)

Ex: $T(v_1) = w_1$

$T(v_2) = 2w_1 - w_2$ - - - - ->

$T(v_3) = 3w_2$

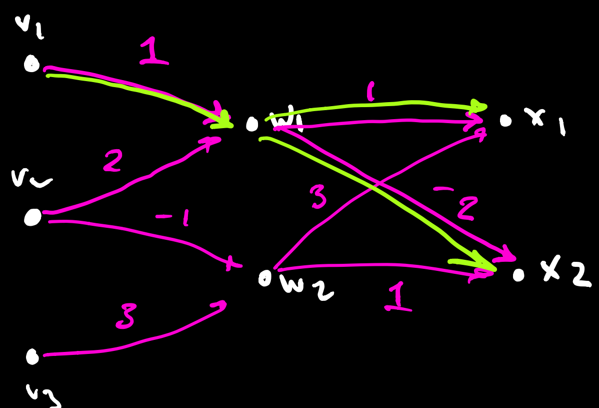


With this visualization, composing matrices is just composing paths.

Ex: $S(w_1) = x_1 - 2x_2$

$S(w_2) = 3x_1 + x_2$

So $T(v_1)$ - - - - ->



FOOD FOR THOUGHT

1. If we choose different bases, how do we visualize this in terms of paths?

2. A matrix is <u>symmetric</u> if $a_{ij} = a_{ji}$.

   What does this mean in terms of paths?

SHIFT GEARS (3.D)

Recall: the identity map $V \longrightarrow V$ sends $v \mapsto v$.

$\mathcal{L}$ script L.

<u>Defn</u> A linear map $T \in \mathcal{L}(V, W)$ is <u>injective</u> if $\forall\ v \neq w$, $T(v) \neq T(w)$.

$T \in \mathcal{L}(V, W)$ is <u>surjective</u> if $\forall\ w \in W\ \exists\ v \in V$ with $T(v) = w$.

$T \in \mathcal{L}(V, W)$ is an <u>isomorphism</u> if it is both injective and surjective.

Ex $P(\mathbb{R}) \to P(\mathbb{R})$ is injective but not surjective.
$p \mapsto x^2 p$

$F^\infty \to F^\infty$ is surjective but not injective.
$(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$

$\mathbb{R}^2 \to \mathbb{R}$ is surjective but not injective.
$(x, y) \mapsto x$

$F^n \to F^n$ is an isomorphism.
$v \mapsto a \cdot v, \quad a \neq 0$