① G-S II

② Approximating functions via polynomials

③ Orthogonal complements.

1) Gram-Schmidt process

Claim: a basis $\{f_1, \dots, f_n\}$ can transform into an orthonormal basis $\{e_1, \dots, e_n\}$, $\boxed{\text{where } \operatorname{span}(f_1, \dots, f_k) = \operatorname{span}(e_1, \dots, e_k)}$

$$e_i' = f_i - \langle f_i, e_1 \rangle e_1 - \dots - \langle f_i, e_{i-1} \rangle e_{i-1}.$$

$$e_i = e_i' / \|e_i'\|.$$

Defined inductively.

<u>Proof</u>. Base case: $\{e_1\}$ orthonormal list by definition.

$\operatorname{span}\{e_1\} = \operatorname{span}\{f_1\}$ by definition.

Inductive step: Assume $\{e_1, \ldots, e_{k-1}\}$ is orthonormal and $\operatorname{span}\{e_1, \ldots, e_{k-1}\} = \operatorname{span}\{f_1, \ldots, f_{k-1}\}$.

$\|e_k\| = 1$ by construction.

$$\langle e_k, \underset{\substack{\uparrow \\ 1 \leq i \leq k-1}}{e_i} \rangle = \left\langle \frac{e_k'}{\|e_k'\|}, e_i \right\rangle = \frac{1}{\|e_k'\|} \langle e_k', e_i \rangle$$

$$= \frac{1}{\|e_k'\|} \langle f_k - \langle f_k, e_1 \rangle e_1 - \ldots - \langle f_k, e_{k-1} \rangle e_{k-1}, e_i \rangle$$

$$= \frac{1}{\|e_k'\|} \left[\langle f_k, e_i \rangle - \underbrace{\sum_{j=1}^{k-1} \langle f_k, e_j \rangle \langle e_j, e_i \rangle} \right]$$

$$= \frac{1}{\|e_k'\|} \left[\langle f_k, e_i \rangle - \langle f_k, e_i \rangle \right]$$

$$= \frac{0}{\|e_k'\|} = 0.$$

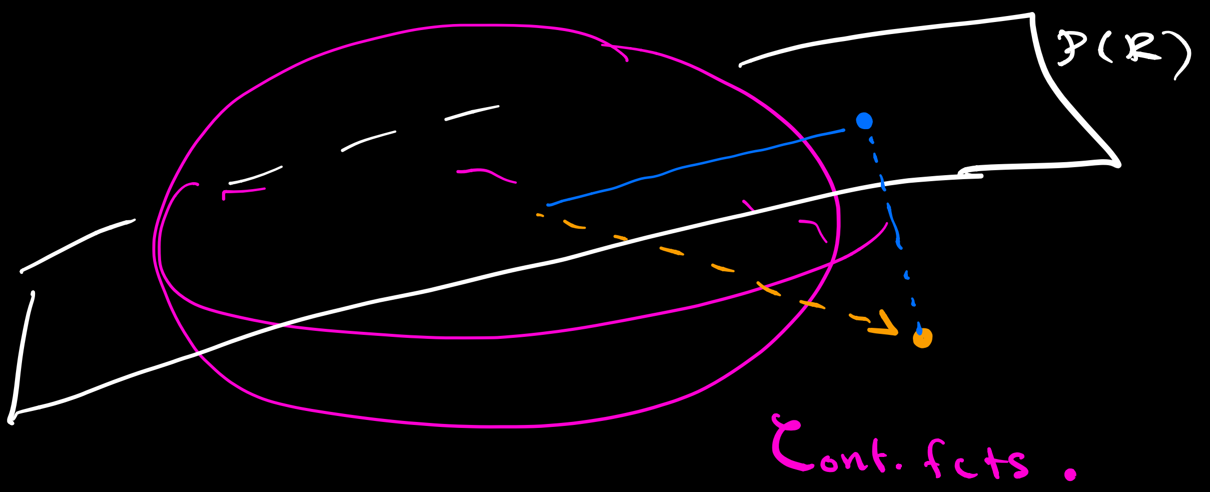
$\operatorname{span}(f_k) \subseteq \operatorname{span}(e_1, \dots, e_k)$ by construction

By induction hypothesis, $\operatorname{span}(f_1, \dots, f_k) \subseteq \operatorname{span}(e_1, \dots, e_k)$.

$\{f_1, \dots, f_k\}$ is l.i. by assumption.

$\{e_1, \dots, e_k\}$ is l.i. by orthogonality.

$\Rightarrow \operatorname{span}(f_1, \dots, f_k) = \operatorname{span}(e_1, \dots, e_k)$. $\square$



Recall Prop: If $\{e_1, \dots, e_k\}$ is an orthonormal list and $v \in V$ then $v_0 = \sum_{i=1}^{k} \langle v, e_i \rangle e_i$ is the closest point on $\text{span}\{e_1, \dots, e_k\}$ to v.

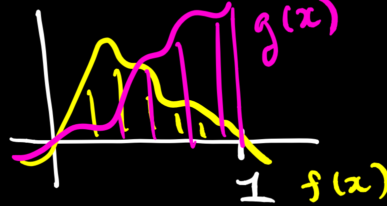
To find polynomial approx. to a function. All we have to do is find an orthonormal basis of $P(\mathbb{R})$ and compute v_0.

Problem Find a degree 2 polynomial approximation to $\sin(x)$. on $[0,1]$

A basis for $P_2(\mathbb{R})$ is $\{\underset{f_1}{1}, \underset{f_2}{x}, \underset{f_3}{x^2}\}$.

The inner product on $\mathcal{C}([0,1]) \supseteq P_2(\mathbb{R})\big|_{[0,1]}$ is given by

$$\int_0^1 f(x)g(x)\,dx = \langle f, g\rangle$$



$$e_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{\int_0^1 1\cdot 1\,dx}} = 1.$$

$$e_2' = x - \langle x, 1\rangle 1 = x - \int_0^1 x\,dx = x - \frac{1}{2}.$$

$$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{x-\frac{1}{2}}{\sqrt{\int_0^1 (x-\frac{1}{2})^2\,dx}} = \sqrt{12}\left(x-\frac{1}{2}\right)$$

$$e_3' = x^2 - \langle x^2, 1\rangle 1 - \left\langle x^2, \sqrt{12}\left(x-\tfrac{1}{2}\right)\right\rangle \sqrt{12}\left(x-\frac{1}{2}\right)$$

$$= x^2 - x + \frac{1}{6}$$

$$e_3 = \frac{e_3'}{\|e_3'\|} = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)$$

The polynomial of degree 2, Q, closest to $\sin(x)$ on $[0,1]$

is $$Q = \langle \sin(x), 1\rangle 1 + \langle \sin(x), \sqrt{12}(x-\tfrac{1}{2})\rangle \sqrt{12}(x-\tfrac{1}{2})$$

$$+ \langle \sin(x), \sqrt{180}(x^2-x+\tfrac{1}{6})\rangle \sqrt{180}(x^2-x+\tfrac{1}{6})$$

$$\approx -.0074 + 1.0913x - 0.2355x^2$$

Taylor Series: $\sin(x) \approx x - \frac{x^3}{6} + \ldots$

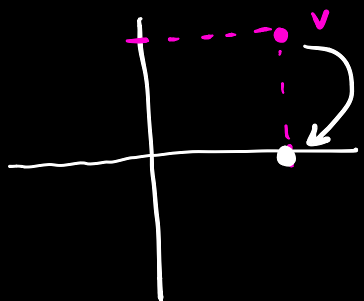
What are we doing?

We are orthogonally projecting.

In general, V has many decompositions $U_1 \oplus U_2 \oplus \dots \oplus U_n$.

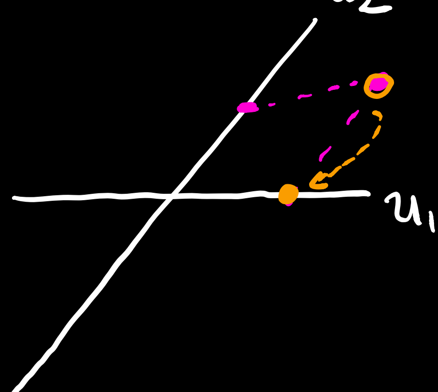
e.g. $\mathbb{R}^2 = \operatorname{span}((0,1)) \oplus \operatorname{span}((1,0))$

$v = (u, w) \longmapsto u$



orthogonal projection map

e.g. $\mathbb{R}^2 =$



$= U_1 \oplus U_2$

$v = u_1 + u_2 \longmapsto u_1$

is a projection map.

Def. Let V be an inner product space with inner product $\langle \, , \, \rangle$ and $S \subseteq V$ be a subset.

Define $S^{\perp} = \{ v \in V \mid \langle v, s \rangle = 0 \;\; \forall \; s \in S \}$

i.e. $S^{\perp}$ is all vectors in V that are orthogonal to all vectors in S.

"S perp"

short for perpendicular.

Prop V an inner product space

(1) $S \subseteq V$ subset, $S^{\perp}$ is a subspace

(2) " ", then $S \cap S^{\perp} = \{0\}$

(3) $S_1, S_2 \subseteq V$ subsets, $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

(4) $S \subseteq V$ subset, $S^{\perp} = \operatorname{span}(S)^{\perp}$.

everything above is repeated.

Pf. (1) $\{0\} \in S^{\perp}$ because $\langle 0, s\rangle = 0$ by properties of $\langle , \rangle$.

If $u, v \in S^{\perp}$ and $s \in S$ then

$$\langle u+v, s\rangle = \langle u, s\rangle + \langle v, s\rangle \quad \text{by prop. of } \langle , \rangle.$$

$$= 0 + 0$$

$$= 0.$$

$\Rightarrow u+v \in S^{\perp}$.

If $u \in S^{\perp}$, $\lambda \in F$, $s \in S$ then

$$\langle \lambda u, s\rangle = \lambda \langle u, s\rangle \quad \text{by prop. of } \langle , \rangle$$

$$= \lambda \cdot 0$$

$$= 0$$

$$\Rightarrow \lambda u \in S^{\perp}.$$

(2) $u \in S \cap S^{\perp} \Rightarrow u \in S$ and $u \in S^{\perp}$

by def., $\langle u, s \rangle = 0 \quad \forall \ s \in S$

in particular, $\langle u, u \rangle = 0$

$\Rightarrow u = 0.$

(3) $v \in S_2^{\perp}$. So $\langle v, u \rangle = 0 \quad \forall \ u \in S_2.$

If $S_1 \subseteq S_2$, then $u \in S_1 \Rightarrow u \in S_2$

So $\langle v, u \rangle = 0 \ \forall \ u \in S_1.$

$\Rightarrow v \in S_1^{\perp}.$

(4) Have to show: $S^\perp \subseteq \text{span}(S)^\perp$ and

$$\text{span}(S)^\perp \subseteq S^\perp.$$

By def., $S \subseteq \text{span}(S)$.

By (3), $\text{span}(S)^\perp \subseteq S^\perp$.

Let $v \in S^\perp$.

Let $u \in \text{span}(S)$. Write $u = \lambda_1 s_1 + \ldots + \lambda_n s_n$,

$\lambda_i \in F$, $s_i \in S$.

Then $\langle v, u\rangle = \langle v, \lambda_1 s_1 + \ldots + \lambda_n s_n\rangle$

$= \bar{\lambda}_1 \langle v, s_1\rangle + \ldots + \bar{\lambda}_n \langle v, s_n\rangle$

$= \bar{\lambda}_1 \cdot 0 + \ldots + \bar{\lambda}_n \cdot 0$

$= 0.$

$\Rightarrow \; v \in \operatorname{span}(S)^{\perp}.$

$\Rightarrow \; S^{\perp} \subseteq \operatorname{span}(S)^{\perp}.$ $\square$