LEAF WISE INTERSECTION POINTS IN NEGATIVE LINE BUNDLES

1. (Old) History / Motivation
2. Using Floer theory (exact)
3. Using Floer theory (monotone)

1. MOTIVATION

Moser '78 studied contact hypersurfaces

$$(\Sigma, \alpha = \vartheta|_{\Sigma}) \subset (M, \omega = d\vartheta)$$

Interested in studying interplay between the Reeb vector field R_α $\left[\begin{array}{l} \alpha(R_\alpha) = 1 \\ d\alpha(R_\alpha, \cdot) = 0 \end{array}\right]$

and other symplectic dynamics.

In particular, R_α spans a 1-D foliation

Call L_x the leaf of this foliation through $x \in \Sigma$,

MOSER'S QUESTION: (Q')

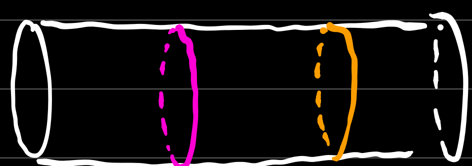
For a fixed symplectomorphism $\varphi : M \to M$ does there exist $x \in \Sigma$, with $\varphi(x) \in L_x$?

Clearly false in general.

e.g. Can displace a curve in $\mathbb{R}^2$ with translation

Is there a more specific question to ask?

1. Is Q' true for non-$\mathbb{R}^{2n}$? T^*S^1 ?



2. Is Q' true for φ a compactly supported Hamiltonian diffeomorphism?

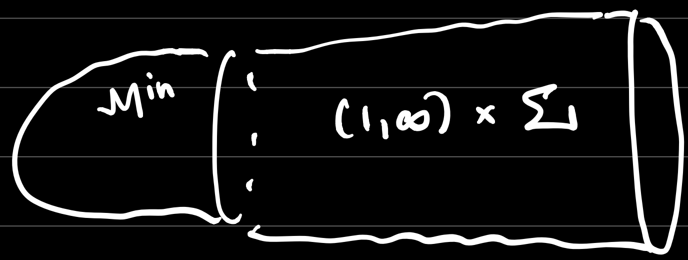
Seems hard to prove in general.

One direction: use Floer theory a la Albers - Frauenfelder.

<u>Thm</u> (A-F) If the Rabinowitz Floer homology $RFH^{\bullet}(\Sigma, M)$ is non-vanishing, then for any $\varphi \in Ham_c(M)$ Σ contains a leaf wise intersection point.

$$\left.\begin{array}{l} \text{- } M \text{ exact} \\ \text{- } \Sigma \text{ restricted contact type} \end{array}\right\} (*)$$

$M = M^{in} \cup (1, \infty) \times \Sigma_1$



<u>Note</u>: $(*) \Rightarrow (0, \infty) \times \Sigma_1 \subset M$

2. Floer theory (exact case)

$H : M \times S^1 \to \mathbb{R}$.

$\rightsquigarrow$ X_H the v.f. defined by $dH(\cdot) = \omega(\cdot, X_H)$.

$\rightsquigarrow$ periodic orbits of φ_H.

Observation: we can use certain H to study the contact dynamics of Σ_1.

Let H_τ be C^2-small in M^{in}

grow like $\tau \cdot R$ on $(1, \infty)_R \times \Sigma_1$

then the periodic orbits of H_τ look like

$\{$ Morse critical points in $M^{in}\}$ $\cup$

$\{$ Reeb orbits of p.d. $< \tau$ in

$(1, \infty) \times \Sigma\}$

[Exercise: by direct calculation]

Organize periodic orbits into a chain cplx.

$$CF^{\cdot}(H_\tau) = \bigoplus_{\dot{x} = X_H} \mathbb{K}\langle x \rangle$$

Take the direct limit:

$$SC^{\cdot}(M) = \varinjlim_{\tau} CF^{\cdot}(H_\tau)$$

$SC^{\cdot}(M)$ encodes data about all Reeb orbits. For example, if

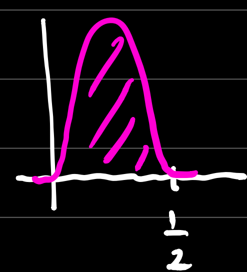
$$H[SC^{\cdot}(M)] \neq H_{morse}(M)$$

then Σ contains at least one Reeb orbit.

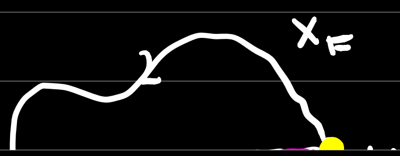
INCREDIBLY ROBUST

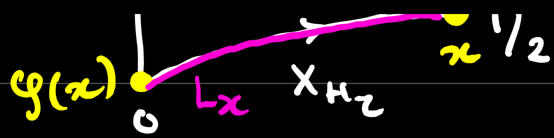
Suppose I want to understand the leafwise $\pitchfork$ pts. of some $F : M \times [1/2, 1] \to \mathbb{R}$ where F is compactly supported and $F(1/2) = F(1) = 0$.

Consider $\mathcal{H}_t = F + \underbrace{\rho(t)} \cdot H_t$



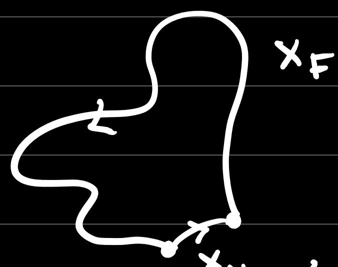
The orbits of $\mathcal{H}_t$ look like:





If X_{H_t} is proportional to R_α then this periodic orbit exhibits a leaf wise $\pitchfork$ pt.

In M^{in}, these orbits look like



X_{H_t} <- not proportional to R_α.

<u>Surprising Thm</u> (Mash-up Albers-Frauenfelder, Cieliebak-Frauenfelder-Oancea, Ritter)

If $H[SC^\cdot(M)] \neq 0$ then $\{r\} \times \Sigma$ has a leaf wise intersection point $\forall r$

and $\forall \varphi \in \mathrm{Ham}_c(M)$.

Examples (1) Surfaces punctured >1 time

(2) $S^*M \subset T^*M$.

3. MONOTONE SYMPLECTIC MANIFOLDS

e.g. $\mathcal{O}(-1) \to \mathbb{C}P^1 \supset (1,\infty) \times S^3$

but doesn't contain any more of the symplectization.

e.g. $\mathcal{O}(-m) \to \mathbb{C}P^n \quad 1 \leq m \leq n$

Thm (Ritter) If $(M, \Omega) \to (B, \omega)$ is a toric, $\overbrace{\underbrace{\text{monotone}}}^{c>0}$, $\underbrace{\text{negative}}_{k>0}$ line bundle, then $H[SC^\cdot(M)] \neq 0$.

What does this tell us about leaf wise $\pitchfork$ pts? Unfortunately, nothing.

However, there is a more refined invariant that we can study.

Take $\Lambda = \mathbb{K}((T))$

Define H_τ^R to be C^2-small on the disk bundle of radius R and grow like $\tau \cdot r$ on its complement.

Define $\mathcal{A}_{H_\tau^R}(T^a x) = a - \int_D \tilde{x}^* \Omega + \int_0^1 H(x)\, dt$

Λ p.d. orbit

$\tilde{x}$ is the fiber disk bounding x.

$\mathcal{A}$ induces a filtration on $\varinjlim_\tau CF^\cdot(H_\tau^R)$.

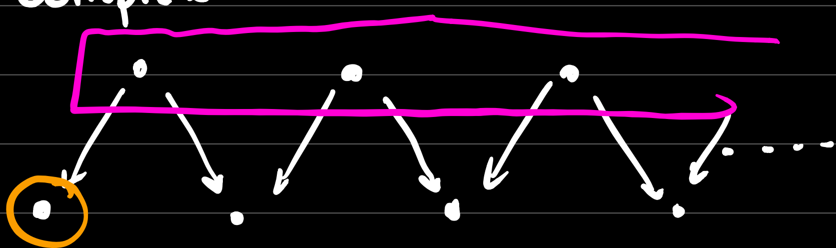
$$SC_a^\cdot(H^R) = \varinjlim CF^\cdot(H_\tau^R) \Big/ \langle \mathcal{A}(T^a x) > a \rangle$$

Taking the inverse limit,

$$\widehat{SC}^{\cdot}(H^R) = \varprojlim_{a} SC^{\cdot}_{a}(H^R)$$

"formal sums whose action $\to \infty$".

This 'completing' can drastically change a chain complex.



$$\mathbb{K} = \mathbb{Z}_2.$$

And indeed,

if $R < 1/\sqrt{k\pi c}$, $H[\widehat{SC}^{\cdot}(H^R)] = 0$

if $R > 1/\sqrt{k\pi c}$, $H[\widehat{SC}^{\cdot}(H^R)] =$

$$H[SC^{\cdot}(H^R)] \neq 0.$$

Thm (V) The radius - $1/\sqrt{k\pi c}$ circle bundle has a leaf wise $\pitchfork$ pt. for any $\varphi \in Ham_c(M)$.

"**Proof**". Take R "just smaller" than $1/\sqrt{k\pi c}$. Use robustness to replace H_c^R with $\mathcal{H}_c^R$:

$$H[\widehat{SC}^\cdot(H^R)] \cong H[\widehat{SC}^\cdot(\mathcal{H}^R)].$$

If Σ has no leaf wise $\pitchfork$ pts, then $\widehat{SC}^\cdot(\mathcal{H}^R)$ is finitely generated / Λ.

So: $\widehat{SC}^\cdot(\mathcal{H}^R) = SC^\cdot(\mathcal{H}_R)$.

$$0 = H[\widehat{SC}^\cdot(H_c^R)]$$

$$= H[\widehat{SC}^\cdot(\mathcal{H}^R)]$$

$$= H[SC^{\cdot}(\mathcal{H}^{R})]$$

$$= H[SC^{\cdot}(H^{R})]$$

$$\neq 0. \qquad \square$$

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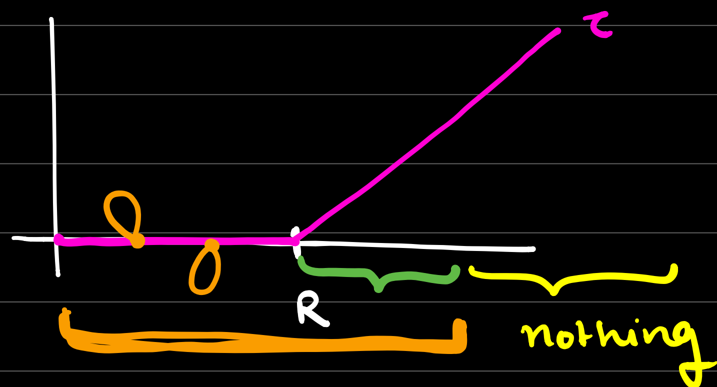
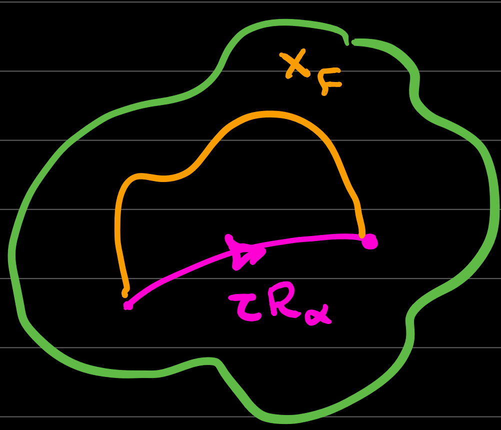




$\boxed{\{\text{p.d. orbits of } F \text{ inside } D_R\}} \cup$

$\{\ell.\text{w.h. pts in the complement of } D_R\}$

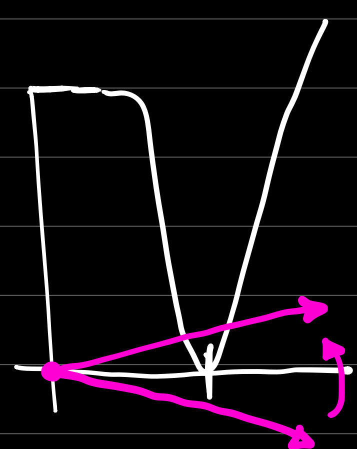
$\varinjlim_{z} CF^{\cdot}(H_z)$

② Does an action filtration tell me anything?

{um} consider, for the heck of it,

a chain complex completed w.r.t.

action filtration. $\left(\varprojlim_{a} \right)$



$$\begin{tikzcd} SC_* \arrow[rr, "c"] & & SC^* \arrow[dl] \\ & \mathrm{Cone}(c) \arrow[ul] & \end{tikzcd}$$

"="

SC · v-shaped.