TODAY

How to build intuition: a sketch

Axler 3.D

HW 2 #5 Let V be a v.s. over F

Let V^* be functions from V to F that satisfy

$$f(v+w) = f(v) + f(w) \quad \forall\ v, w \in V$$

$$f(a \cdot v) = a \cdot f(w) \quad \forall\ v \in V,\ a \in F.$$

Find a basis for V^*.

FIRST STEP: WORK OUT SIMPLEST "NON-TRIVIAL" EXAMPLE

For me: $V = F$

$\mathbb{R}$

Start making observations...

① $f(v+0) = f(v) + f(0) \rightsquigarrow \boxed{f(0) = 0}$

$\quad \| $

$\quad f(v)$

② $f(a \cdot 1) = a \cdot f(1)$

(3) $\{v\}$ is a basis for F, $f(a\cdot v) = a\cdot f(v)$.

(4) f is determined by its behavior on v.

(5) If $f(v) = 1$ and $g(v) = a$ then $\boxed{g(v) = a\cdot f(v)}$

g and v are linearly dep.

<u>SECOND STEP</u>] Understand what the problem is ⇓ asking. What does it mean for $\{f_1, \dots, f_n\}$ to be a basis?

$\{f\}$ is a basis.

i) linear independence:

$$a_1 f_1 + \dots + a_n f_n = \underbrace{0} \quad \text{then} \quad a_1 = \dots = a_n = 0$$

the 0 element is the function $z(v) = 0$ $\forall v \in V$.

$$\forall v \in V, \quad \underbrace{a_1 f_1(v) + \dots + a_n f_n(v)} = z(v) = 0.$$

ii) spanning: $\forall\ g \in V^*$ $\exists\ a_1, \dots, a_n \in F$ such that

$$g(v) = a_1 f_1(v) + \dots + a_n f_n(v) \quad \forall\ v \in V$$

Q Can I use this info. and combine it with my previous observations.

Talk to people.

THIRD STEP / Slightly harder example. $V = F^2$

① If $\{v_1, v_2\}$ is a basis for F^2, then

$$f(a_1 v_1 + a_2 v_2) = a_1 f(v_1) + a_2 f(v_2).$$

② f is determined by v_1 and v_2

? ?

③ 2 conditions $\Rightarrow$ V^* is (?)

2-dimensional

(4)

$$\begin{tikzcd} F \arrow[dr, squiggly, "f"] & \\ \oplus & F \\ F \arrow[ur, squiggly, "f"'] & \end{tikzcd}$$

"Looks like I'm defining 2 separate fcts".

(5) How does my work from Step 1 fit in?

I have a function

$f_1(v_1) = 1$ anything nonzero.

$f_1(v_2) = 0$

(6) I also have a function

$f_2(v_1) = 0$

$f_2(v_2) = 1$ anything non-zero.

Requires audacity! To say "let me check if $\{f_1, f_2\}$ is a basis".

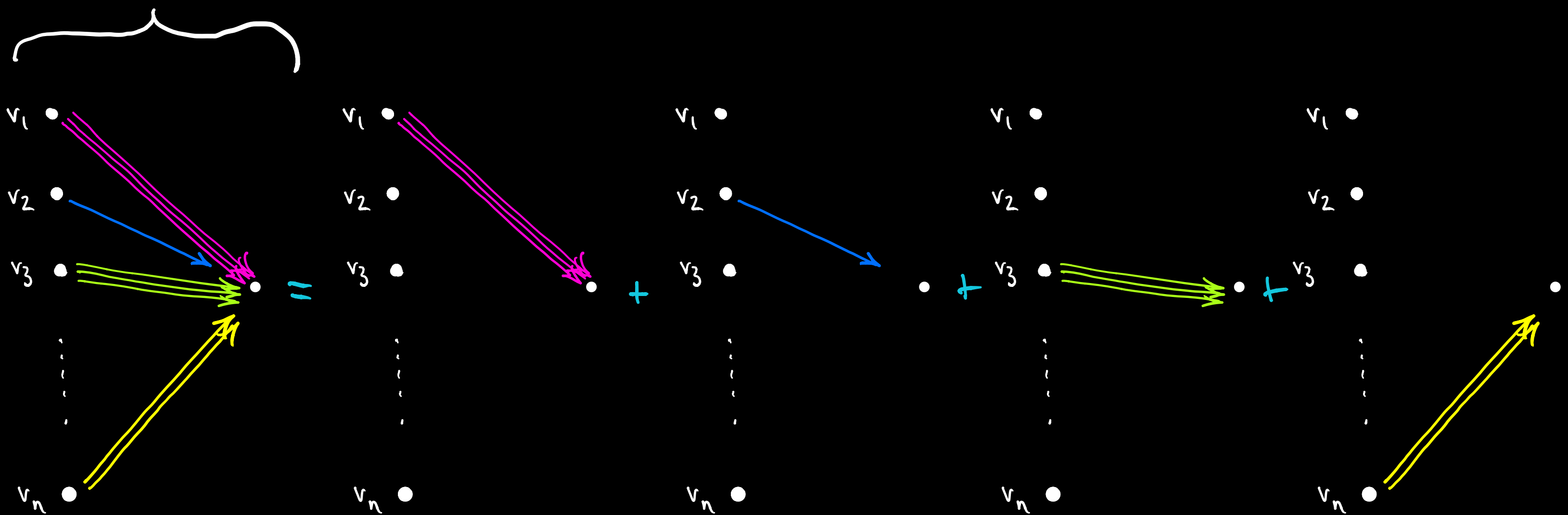
STEP 4 I see a pattern.... if I have a basis $\{v_1, \ldots, v_n\}$ for V then maybe $\{f_1, \ldots, f_n\}$ is a basis for V^*,

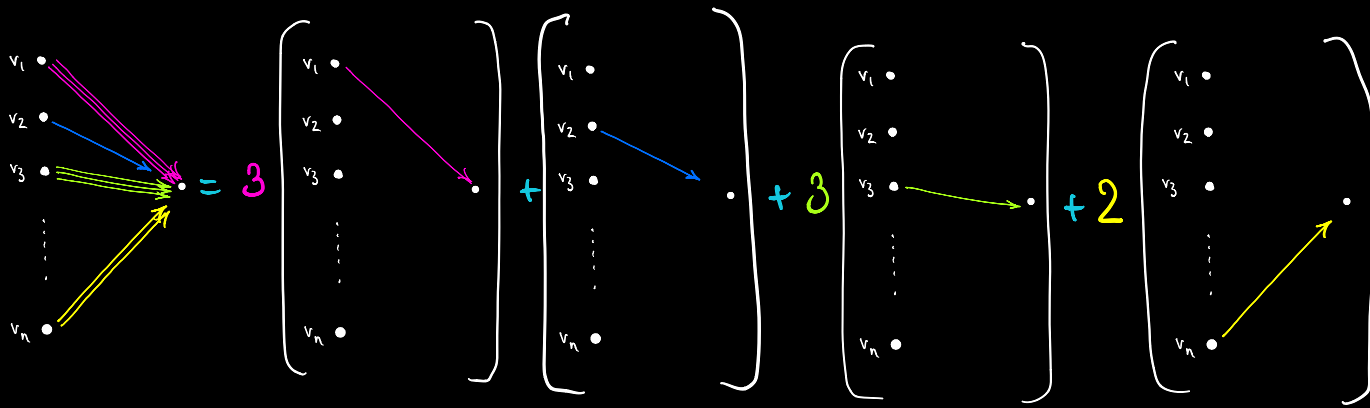
where
$$f_i(v_i) = 1$$
or anything non-zero!
$$f_i(v_j) = 0 \qquad j \neq i.$$

STEP 5 Go back and see if the intuition that I had still makes sense, and is still "cleanest".

Could I have visualized this?

$$f(a_1 v_1 + \dots + a_n v_n) = \underset{\substack{\shortparallel \\ 3}}{a_1} f(v_1) + \underset{\substack{\shortparallel \\ 1}}{a_2} f(v_2) + \underset{\substack{\shortparallel \\ 3}}{a_3} f(v_3) + \underset{\substack{\shortparallel \\ 2}}{a_n} f(v_n)$$




v1
v2
v3
vn
= 3
+
+ 3
+ 2

$$\begin{bmatrix} 3 & 1 & 3 & 0 & \cdots & 0 & 2 \end{bmatrix} =$$

$$3\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} +$$

$$1\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} +$$

$$3\begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} +$$

$$2\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Recall a result from last time:

(3) $\dim(V) = \dim(W)$, $T \in \mathcal{L}(V, W)$ then T is

injective $\iff$ T is surjective.

bijective

So (3) can be rephrased: T is injective $\iff$ T is surjective $\iff$ T is bijective.

When do $\underbrace{\text{bijective linear transformations}}_{\text{isomorphism.}}$ exist?

Prop. If V and W are finite-dimensional vector spaces over a field $\mathbb{F}$ then an isomorphism $T: V \to W$ exists $\iff \dim(V) = \dim(W)$.

Pf. ($\Leftarrow$) Suppose $\dim(V) = \dim(W)$.

Then a basis $\{v_1, \dots, v_n\}$ of V has the same size as a basis $\{w_1, \dots, w_n\}$ of W.

Fix two such bases.

Define $T: V \longrightarrow W$ and extend linearly.

$$v_i \longmapsto w_i$$

By construction, T is linear.

T is injective: let $v = a_1 v_1 + \dots + a_n v_n$ and

let $v' = b_1 v_1 + \dots + b_n v_n$

Suppose $T(v) = T(v')$.

Then $T(a_1v_1 + \ldots + a_nv_n) = T(b_1v_1 + \ldots + b_nv_n)$

$$a_1T(v_1) + \ldots + a_nT(v_n) = b_1T(v_1) + \ldots + b_nT(v_n)$$

$$a_1w_1 + \ldots + a_nw_n = b_1w_1 + \ldots + b_nw_n$$

$$(a_1 - b_1)w_1 + \ldots + (a_n - b_n)w_n = 0.$$

Since $\{w_1, \ldots, w_n\}$ is a basis, $a_i = b_i \ \forall i$, and so $v = v'$.

T is surjective: for $w \in W$, write $w = a_1w_1 + \ldots + a_nw_n$

By linearity, $T(a_1v_1 + \ldots + a_nv_n) = w$.

($\Rightarrow$) Conversely, suppose $T: V \to W$ is an isomorphism.

T is injective, and so $\mathrm{null}(T) = \{0\}$

T is surjective, and so $\mathrm{im}(T) = W$.

Rank-nullity Thm: $\dim(V) = \dim(\mathrm{null}(T)) + \dim(\mathrm{im}(T))$

$$= 0 + \dim(W)$$

$$\dim(V) = \dim(W)$$

$\square$