WHAT IS THIS CLASS ABOUT??

Proof-based

Linear Algebra

WHAT IS A PROOF?

“evidence or argument establishing or helping to establish a fact or the truth of a statement.” -Google Dictionary

“able to withstand something damaging; resistant.” -Google Dictionary

"The right wing ("right-or-wrong", "rule-of-law") definition is that a proof is a logically correct argument that establishes the truth of a given statement. The left wing answer (fuzzy, democratic, and human centered) is that a proof is an argument that convinces a typical mathematician of the truth of a given statement.” -Prof Devlin (@Stanford)

WHEN IS A PROOF?

WHY IS A PROOF?

https://youtu.be/sS-x_NQ-CsI

Think about: what math buzz-words or buzz-concepts do they use to sell their product?

1. Precision
2. Robustness
3. Efficiency.

LINEAR ALGEBRA

Central Objects of Study

- Vector spaces ← objects
- Linear maps ← maps ← functions

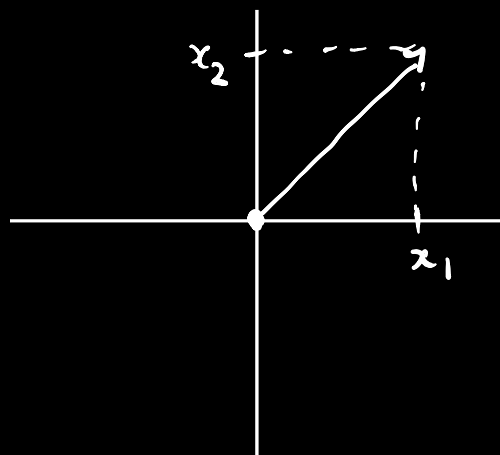
Objectives for today

- Define a vector space over the real numbers.
- Give examples of real vector spaces.

VECTOR SPACES

Example: $\mathbb{R}^2 = \{$set of ordered pairs $(x_1, x_2) = \vec{x}$, where x_1, x_2 are real numbers$\}$ AKA Euclidean plane.

Geometrically $\vec{x}$ represents a vector with tail $(0,0)$ and head (x_1, x_2)



We write $\vec{x}$ or $\boldsymbol{x}$

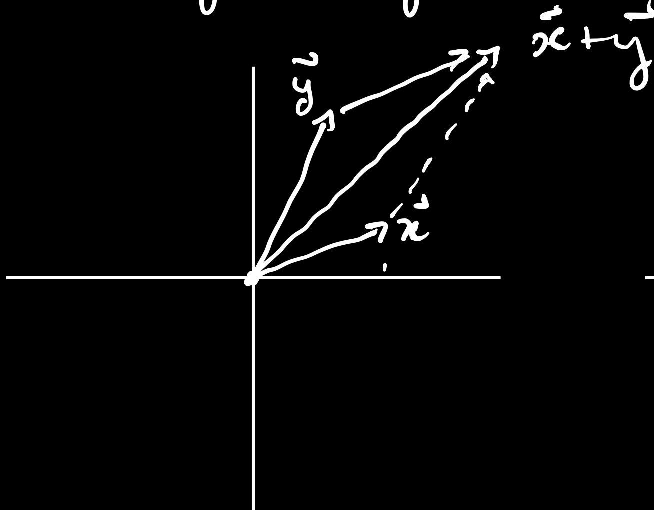
Linear algebra studies 2 algebraic properties of vectors and vector spaces: _vector addition_ and _scalar multiplication_.

If $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ are vectors in $\mathbb{R}^2$, then

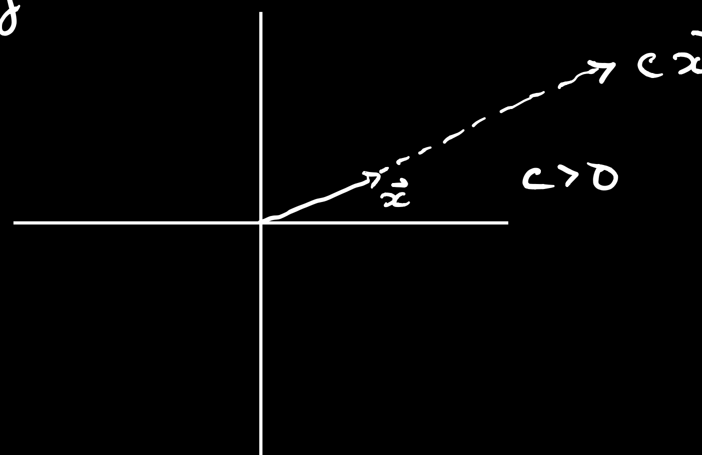
$$\vec{x} + \vec{y} = (x_1 + y_1, x_2, y_2)$$

If $\vec{x} = (x_1, x_2)$ a vector in $\mathbb{R}^2$ and c is a real number (a _scalar_) then

$$c \cdot \vec{x} = (cx_1, cx_2)$$



vector addition



scalar multiplication

WHAT ARE THE FUNDAMENTAL PROPERTIES OF VECTOR ADDITION AND SCALAR MULTIPLICATION?

① If $\vec{x}, \vec{y}, \vec{z}$ are vectors in $\mathbb{R}^2$, does $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$?

② If $\vec{x}, \vec{y} \in \mathbb{R}^2$ does $\vec{x} + \vec{y} = \vec{y} + \vec{x}$?

⑤ If $\vec{x}, \vec{y} \in \mathbb{R}^2$ and c is a scalar, does

$$c \cdot (\vec{x} + \vec{y}) = c \cdot \vec{x} + c \cdot \vec{y} \,?$$

⑥ If $c, d \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$, does $(c+d) \cdot \vec{x} = c \cdot \vec{x} + d \cdot \vec{x}$?

⑦ If $c, d \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$, does $c(d \cdot \vec{x}) = (c \cdot d) \cdot \vec{x}$?

Reflect: Do you believe these because of....

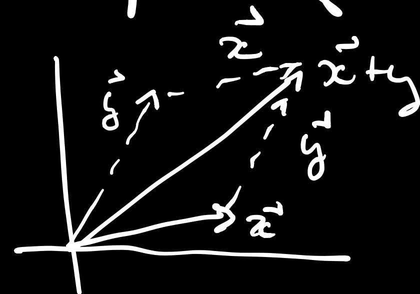
Algebra

or

Geometry

$$\vec{x}+\vec{y} = (x_1+y_1, x_2+y_2)$$
$$= (y_1+x_1, y_2+x_2)$$
$$= \vec{y}+\vec{x}$$

or ....



WHAT ARE THE ALGEBRAIC PROPERTIES OF VECTOR ADDITION AND SCALAR MULTIPLICATION?

(3) There is an additive identity element $\vec{0} \in \mathbb{R}^2$ with the property that for all $\vec{x} \in \mathbb{R}^2$, $\vec{0} + \vec{x} = \vec{x}$?

Q. What is $\vec{0}$? $\vec{0} = (0,0)$

(4) $\forall \vec{x} \in \mathbb{R}^2$ there is an additive inverse element $-\vec{x} \in \mathbb{R}^2$, which has the property that

$$\vec{x} + (-\vec{x}) = \vec{0}.$$

(8) $\forall \vec{x} \in \mathbb{R}^2$, $1 \cdot \vec{x} = \vec{x}$.

THESE ARE THE 8 ALGEBRAIC PROPERTIES OF $\mathbb{R}^2$

they arise from the properties of $\mathbb{R}$

For every $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{R}^2$ and for every c and d in $\mathbb{R}$,

1. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. there is an additive identity $\vec{0}$ such that $\vec{x} + \vec{0} = \vec{x}$
4. there is an additive inverse $-\vec{x}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
5. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
6. $(c + d)\vec{x} = c\vec{x} + c\vec{y}$
7. $c(d\vec{x}) = (cd)\vec{x}$
8. 1 is the multiplicative identity scalar for $\mathbb{R}^2$

uniqueness?

$$\underbrace{0 \cdot \vec{x} = 0} \quad ?$$

WHAT OTHER SPACES HAVE OPERATIONS THAT SATISFY THESE RULES?

1. $\mathbb{R}$

2. $\mathbb{R}^3$

3. $\mathcal{F}(\mathbb{R})$ = set of all functions $\{f : \mathbb{R} \to \mathbb{R}\}$

   a. $f(x) = x^2$

   b. $g(x) = \sin(x)$

   c. $h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

① Is there addition on $\mathcal{F}(\mathbb{R})$?

$$(f+g)(x) = f(x) + g(x)$$

② Is there scalar multiplication on $\mathcal{F}(\mathbb{R})$?

$$(c \cdot f)(x) = c \cdot f(x).$$

Do the 8 properties of $\mathbb{R}^2$ hold for $\mathcal{F}(\mathbb{R})$?

Let's just check commutativity ....

Let $f, g \in \mathcal{F}(\mathbb{R})$. $\forall x \in \mathbb{R}$,

$$(f+g)(x) \underset{\substack{\uparrow \\ \text{defn.}}}{=} f(x) + g(x) \underset{\substack{\uparrow \\ \text{commutativity} \\ \text{of } \mathbb{R}}}{=} g(x) + f(x) = (g+f)(x)$$

✓

DEFINITION: A real vector space is: a set V with

(i) an operation called vector addition, $+$, so that $\forall\ v_1, v_2 \in V$, $v_1 + v_2 \in V$.

(ii) an operation called scalar multiplication, $\cdot$ so that $\forall\ v \in V$ and $c \in \mathbb{R}$, $c \cdot v \in V$.

satisfying

1. $\forall\ v_1, v_2, v_3 \in V$ $\quad v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
2. $\forall\ v_1, v_2 \in V$ $\quad v_1 + v_2 = v_2 + v_1$
3. $\exists$ a zero element $\vec{0} \in V$, s.t. $\vec{v} + \vec{0} = \vec{v}\ \ \forall\ \vec{v} \in V$

④ $\forall\ v \in V\ \exists$ an additive inverse $(-v) \in V$ s.t.

$$v + (-v) = \vec{0}.$$

⑤ If $v_1, v_2 \in V$ and $c \in \mathbb{R}$, then $c \cdot (v_1 + v_2) =$

$$c \cdot v_1 + c \cdot v_2$$

⑥ If $v \in V$ and $c, d \in \mathbb{R}$, then $(c+d) \cdot v = c \cdot v + d \cdot v$

⑦ If $v \in V$ and $c, d \in \mathbb{R}$, then $c \cdot (d \cdot v) = (c \cdot d) \cdot v$

⑧ $\forall\ v \underbrace{\in}_{\text{"in"}} V, \quad 1 \cdot v = v.$

the only elements in V are vectors.

MORE EXAMPLES

$\mathbb{R}^n$, $n \geq 1$.

Is {polynomials of degree n} a vector space? NO

Hint. If $p(x)$ and $q(x)$ are two degree n polynomials, is $(p+q)(x)$ always a degree-n polynomial?

no. $x^2 + x = p(x)$

$x^2 = q(x)$

$(p-q)(x) = x$

However,

$\{$ polynomials of degree $\leq n$ is a vector space $\}$

(Exercise)

Next time: "unreal" vector spaces

Vector spaces over <u>fields</u>

Proving other properties of vector spaces