Thm. If $\mathbb{F} = \mathbb{C}$ and V is a finite-dimensional vector space, then any $T \in \mathcal{L}(V)$ has an eigenvalue.

Idea. By the Prop, all we need to do is find some $\lambda \in \mathbb{C}$ for which the map $T - \lambda \cdot I_V$ is not injective.

If I have a bunch of candidate eigenvalues $\lambda_1, \dots, \lambda_k$ I can test them all at once by checking if the composition

$$(T - \lambda_1 I_V) \circ \underbrace{(T - \lambda_2 I_V) \circ \dots \circ}_{\text{this looks like a factored polynomial.}} (T - \lambda_k I_V)$$

is injective.

If this sends some non-zero $v \in V$ to $\vec{0}$, then at least one of the $T - \lambda_i I_V$ s is not injective.

Idea: let's find a polynomial to factor.

Pf. Let $\dim(V) = n$.

Choose some non-zero $v \in V$.

The list of vectors $\{v, T(v), T^2(v), \ldots, T^n(v)\}$ is linearly dependent. (it has $n+1$ vectors!)

$\leadsto$ There are scalars $a_0, a_1, \ldots, a_n \in \mathbb{C}$, not all zero, with

$$\vec{0} = a_0 v + a_1 T(v) + \ldots + a_n T^n(v).$$

$$= \underbrace{(a_0 I_V + a_1 T + a_2 T^2 + \ldots + a_n T^n)}_{\text{This is a polynomial with coefs in } \mathbb{C}}(v)$$

(in the variable T)

So I can factor it by the Fundamental Thm of Algebra.

$$\vec{0} = \underbrace{\overset{\neq 0}{c} \cdot (T - \lambda_1 I_V)(T - \lambda_2 I_V) \cdots (T - \lambda_n I_V)}_{\text{not injective.}}(v)$$

$\Rightarrow \exists$ some i, with $(T - \lambda_i I_V)(v) = 0$.

$\Rightarrow \exists$ some i, with $T - \lambda_i I_V$ not injective.

By Prop, λ_i is an eigenvalue of T.

$\square$

Cor. T always has a 1-dimensional invariant subspace.

$\boxed{Q}$ ① What are maps $\mathbb{C}^2 \to \mathbb{C}^2$ with only one eigenvalue?

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ the matrix in the basis $\{(1,0),(0,1)\}$ of the map $T(x,y) = (\lambda x, \lambda y)$.

examples of eigenvectors: $(1,0)$
$(0,1)$
$(1,1)$
$\vdots$

ALL VECTORS ARE EIGENVECTORS.

$\boxed{Q}$ What is a map $\mathbb{C}^2 \to \mathbb{C}^2$ where not everything is an eigenvector?

(2)

Suggestion: $(w, z) \mapsto (-z, w)$.

This has 2 eigenvalues: $+i$, $-i$.

As we'll see shortly, this means that there is a basis $\{v_1, v_2\}$ of $\mathbb{C}^2$ for which the matrix of this map is $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

v_1 and v_2 are both eigenvectors.

But, for example, $v_1 + v_2$ is not. (probably)

$T(v_1 + v_2) = i \cdot v_1 - i \cdot v_2 \neq \lambda \cdot (v_1 + v_2)$.

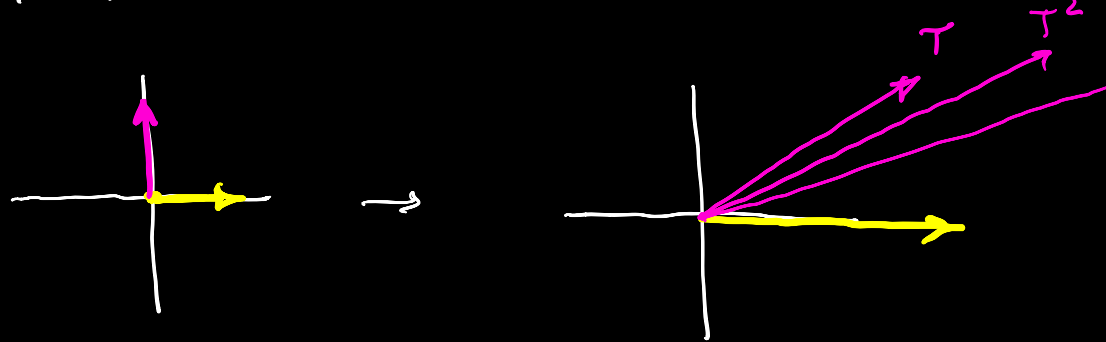
Another example: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Does not have a basis of eigenvectors.

(3)

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x + y \\ \lambda y \end{bmatrix} = \mu \begin{bmatrix} x \\ y \end{bmatrix} \iff \begin{matrix} \mu = \lambda \\ y = 0 \end{matrix}.$$

$\rightsquigarrow$ the only eigenvectors that occur are of the form $(x, 0)$.

in $\mathbb{R}^2$:



Defn. The *eigenspace* of $T \in \mathcal{L}(V)$ corresponding to an eigenvalue λ is the set of vectors

$$\{ v \in V \mid v \in \operatorname{null}(T - \lambda I_V) \}.$$

Ex 1. The eigenspace corresponding to λ is all of $\mathbb{C}$.

Ex 2. The eigenspace corresponding to i is $\operatorname{span}(v_1)$

$-i$ is $\operatorname{span}(v_2)$.

Denote the eigenspace corresponding to λ by $E(\lambda)$.

Prop. If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of $T \in \mathcal{L}(V)$, then the sum $E(\lambda_1) + \ldots + E(\lambda_k)$ is direct.

Pf. If $u_i \in E(\lambda_i)$ and $u_1 + u_2 + \ldots + u_k = 0$ then by the linear independence of eigenvectors with distinct eigenvalues, $u_i = 0 \ \forall i$.

$\square$

DIAGONAL OPERATORS

A diagonal matrix is an $n \times n$ matrix of the form

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

An operator is <u>diagonalizable</u> if there exists a basis for which its matrix is diagonal.

<u>Ex</u>. $T : \mathbb{R}^2 \to \mathbb{R}^2$

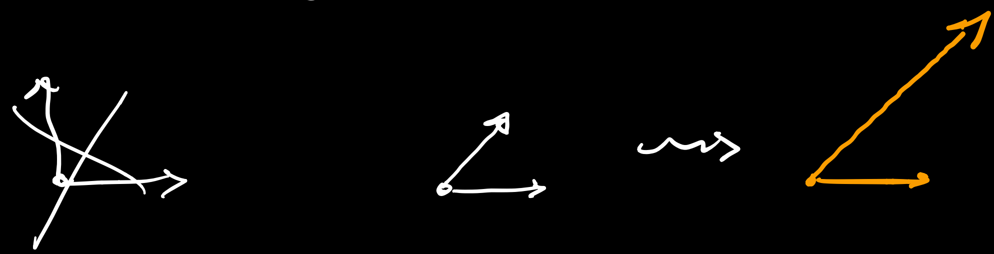
$$(x, y) \mapsto (2x, y+x)$$

$$v_1 = (0,1) \quad T(v_1) = (0,1) = v_1$$

$$v_2 = (1,1) \quad T(v_2) = (2,2) = 2 \cdot v_2$$

In the basis $\{v_1, v_2\}$, the matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

So T is diagonalizable.



Thm If V is finite dim and $T \in \mathcal{L}(V)$ then the following are equivalent:

(1) T is diagonalizable

$\Updownarrow$

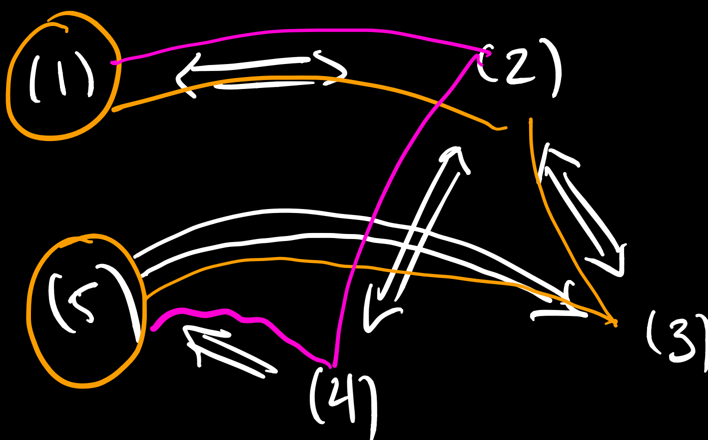
(2) V has a basis consisting of eigenvectors of T.

(3) There exists 1-dimensional subspaces $U_1, \ldots, U_n$, each T-invariant, s.t. $V = U_1 \oplus \cdots \oplus U_n$.

(4) If the distinct eigenvalues of T are $\lambda_1, \dots, \lambda_k$ then $V = E(\lambda_1) \oplus \dots \oplus E(\lambda_k)$.

$\boxed{(5)}$ $\dim(V) = \dim E(\lambda_1) + \dots + \dim E(\lambda_k)$.

"Pf".



Fun fact The graph associated to a diagonal matrix just looks like