So FAR:

Defined adjoints of operators $\boxed{\begin{array}{l} T: V \to W \\ T^*: W \to V \end{array}}$ $\rightsquigarrow$ generalized matrix transpose

self-adjoint operators $\boxed{\begin{array}{c} T: V \to V \\ T = T^* \end{array}}$ generalized symmetric matrices

normal operators: $\boxed{T^*T = TT^*}$ spectral thm.

proved in class only for $F = \mathbb{C}$.

"There is an orthonormal basis for which T is diagonal (T is normal)".

Consider the following problem: Find an operator T_k, whose image has dimension equal to k that best approximates T.

e.g. Suppose T has matrix $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

w.r.t. the standard basis. What is a good candidate for T_1?

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ whose image has dimension 1.

Given by matrix $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$.

e.g. $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix $\begin{bmatrix} 700 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ w.r.t. the standard basis.

Candidate for T_1? $\begin{bmatrix} 700 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Candidate for T_2? $\begin{bmatrix} 700 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

"Best approximation": want a notion of "norm".

So can talk about $\|T - T_k\|$: the "size" of $T - T_k$.

Lots of options! See, for example, Wiki "Matrix norms".

<u>Option 1</u> Treat a matrix like a vector.

Recall: $\boxed{M_{n\times m}(\mathbb{R}) \xrightarrow{\cong} \mathbb{R}^{n\cdot m}.}$

Define $\left\| \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \right\| = \sqrt{\sum |a_{ij}|^2}.$

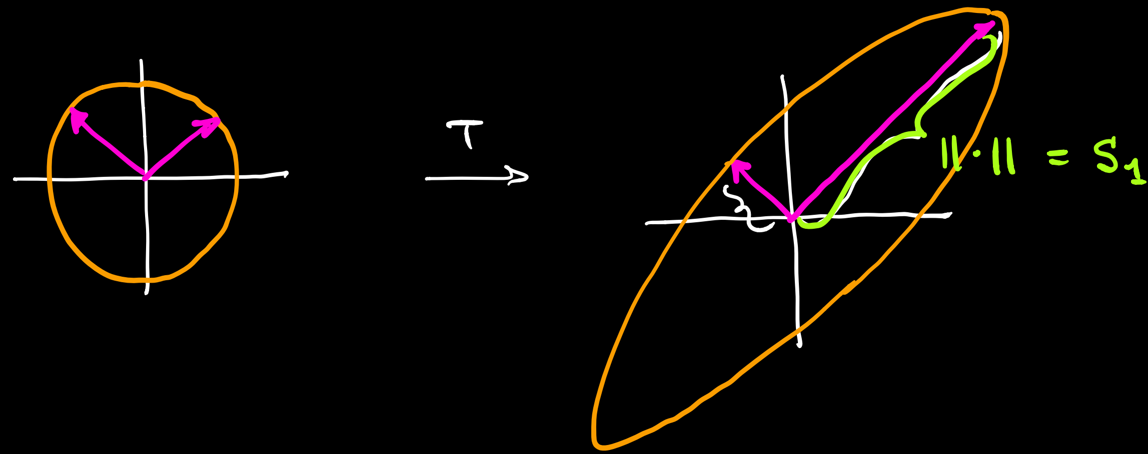
FROBENIUS NORM

e.g. $\left\| \begin{bmatrix} 700 & & 0 \\ & 100 & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} 700 & & 0 \\ & 0 & \\ 0 & & 0 \end{bmatrix} \right\| =$

$$\left\| \begin{bmatrix} 0 & & 0 \\ & 100 & \\ 0 & & 1 \end{bmatrix} \right\| = \sqrt{100^2 + 1^2} \approx 100 = \text{operator norm}$$

Exercise: Frobenius norm is independent of choice of orthonormal basis.

2. "Better norm": **Operator norm**



Define $\|T\| = s_1 = \max_{\substack{v \in V \\ \|v\| = 1}} \|T(v)\|$.

Can rephrase this: $\|T\| = \max_{v \in V} \frac{\|T(v)\|}{\|v\|}$.

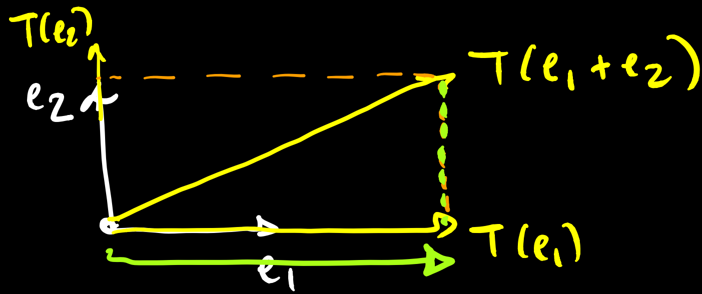
Claim. If $T \in \mathcal{L}(V)$ has an $\boxed{O.}$E.B. with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ then the operator T_k whose matrix w.r.t. the O.E.B. is

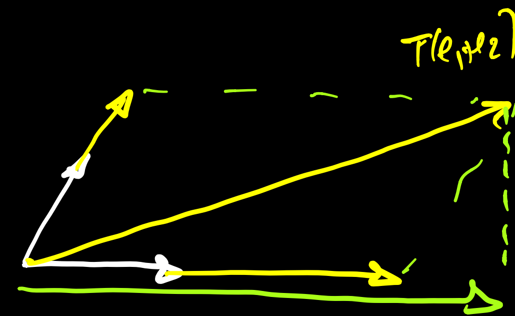
$$\begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_k & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

then $\|T_k - T\| \leq \|S - T\| \quad \forall \quad S \in \mathcal{L}(V)$ whose image has dimension $= k$.

We are writing $T(v) = \sum \lambda_i \langle v, e_i \rangle e_i$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis.





What can we say if T is not self-adjoint or normal?

For example, $T: V \to W$, where $V \ncong W$?

Clever idea: work with two different ortho. bases!

Thm (Singular value decomposition) Let $T \in \mathcal{L}(V, W)$

There exists an orthonormal basis $\{e_1, \dots, e_n\}$ of V and an orthonormal basis $\{f_1, \dots, f_m\}$ of W and scalars $\sigma_1, \dots, \sigma_n \in \mathbb{R}_{\geq 0}$

such that

$$T(e_i) = \sigma_i f_i \quad \forall \quad i \leq \min(n,m)$$

$$T(e_i) = 0 \quad \forall \quad i > \min(n,m)$$

In other words, the matrix of T w.r.t. $\{e\}$ and $\{f\}$ looks like

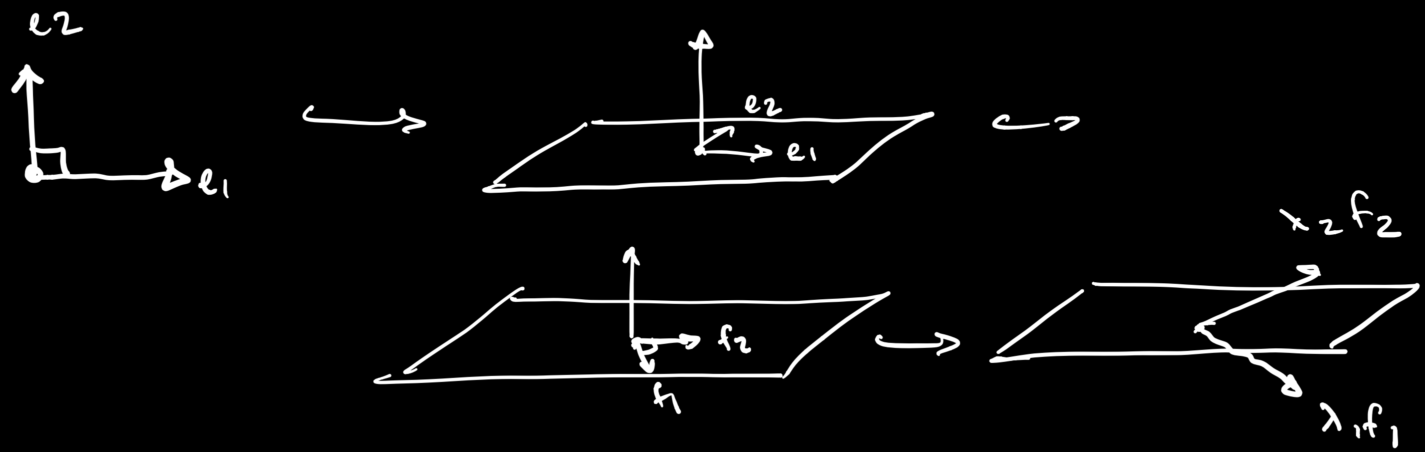
$$\begin{bmatrix} \sigma_1 & & 0 & & & \\ & \ddots & & & & \\ 0 & & \sigma_{\min(n,m)} & 0 & \cdots & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_{\min(n,m)} \\ & & 0 \\ 0 & & \vdots \\ & & 0 \end{bmatrix}$$

In other words, $\boxed{T(v) = \sum_{i=1}^{\min(n,m)} \sigma_i \langle v, e_i \rangle f_i .}$

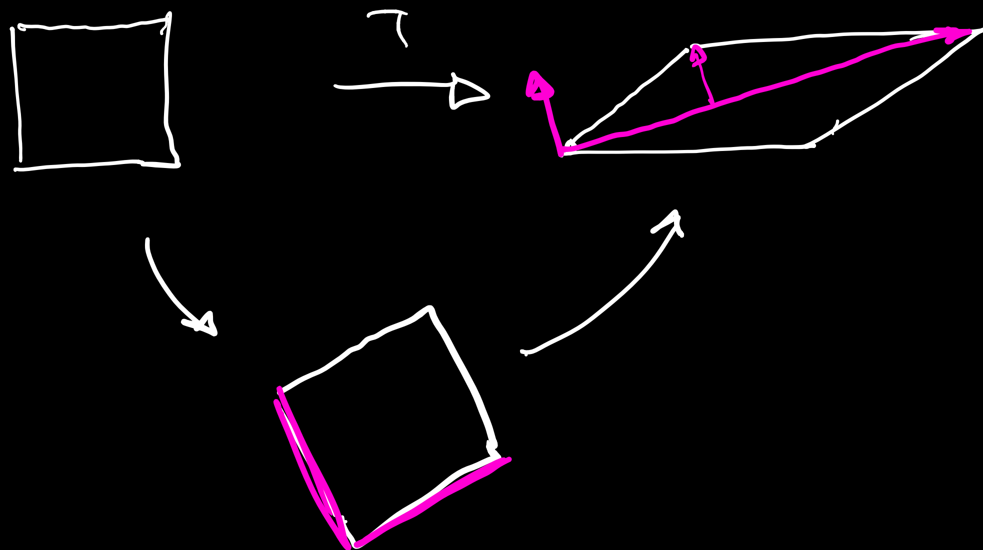
$$T_k(v) = \sum_{i=1}^{k} \sigma_i \langle v, e_i \rangle f_i ,$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma$

T looks like the composition of a projection or inclusion, a rotation, and a stretching.



Another picture (if you remember the tiling principle)



Musing: why $\sigma_i \geq 0$?

e.g. $T : x \mapsto -$ Choose $e_1 = 1$, $e_2 = -1$

then T is the composition $e_1 \mapsto e_2$, $e_2 \mapsto 2e_2$.

Pf of SVD:

NOTE: $T^*T : V \longrightarrow V$

Lemma ~~(To be proven)~~ T^*T is self-adjoint.

defn. of adj, since $(T^*)^* = T$

$$\langle T^*Tv_1, v_2\rangle = \langle Tv_1, Tv_2\rangle$$

$$= \langle v_1, \underbrace{T^*T}v_2\rangle.$$

We conclude that $(T^*T)^* = T^*T$.

$\boxed{\exists \text{ an O.E.B. } \{e_1, \dots, e_n\} \text{ of } T^*T.}$

Set $f_i' = T(e_i)$.

Note: $\langle f_i', f_j' \rangle = \langle Te_i, Te_j \rangle$

$$= \langle e_i, T^*Te_j \rangle$$

$$\propto \langle e_i, e_j \rangle$$

$$= 0.$$

Set: $f_i = \dfrac{f_i'}{\|f_i'\|}$.

Then: $T(e_i) = \underbrace{\|f_i'\|}_{\sigma_i} f_i$.

Note-keeping takes care of subtleties. $\square$