Objectives

1. Check whether a list of vectors is linearly independent
2. Prove that a vector space always has a basis.

"RECALL"

The span$(\vec{v}_1, \vec{v}_2, \ldots)$ is the subspace

$$\{a_1v_1 + \cdots + a_mv_m \mid a_i \in F,\ v_i \in V,\ m \in \mathbb{N}_{>0}\}$$

The direct sum $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_m$ is a sum of subspaces $\mathcal{U}_1 + \cdots + \mathcal{U}_m$ in which $\vec{0}$ is uniquely written.

"Breaking your space up into manageable pieces"

Today

1. Examples of direct sums

⇓ without proof?

2. Linear independence

3. Define basis

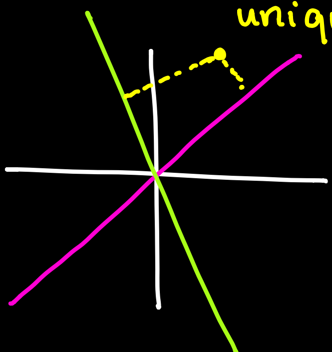
4. Prove a result about finite bases.

$\boxed{0}$ Examples of direct sums

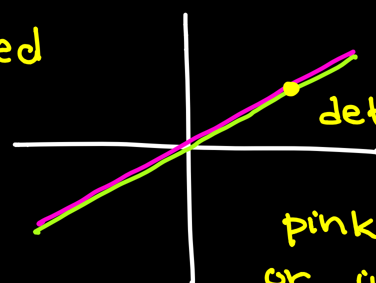
1. $\operatorname{span}(x) + \operatorname{span}(x^2+x) \subset \mathcal{P}_2(\mathbb{R})$ is a direct sum.

2. $\mathcal{U}_1, \mathcal{U}_2$ both lines in $\mathbb{R}^2$ through $\vec{0}$.

   $\mathcal{U}_1 + \mathcal{U}_2$ is a direct sum $\iff \mathcal{U}_1 \neq \mathcal{U}_2$



uniquely determined



determined by looking @ just pink line, or just the green line (multiple ways)

3. Take $V = C^0([-1,1]) := \{\text{continuous functions } [-1,1] \to \mathbb{R}\}$

Exercise: V is a vector space over $\mathbb{R}$.

Take $\mathcal{U}_1 = \{ f \in V \mid f(x) = f(-x) \}$ ← even functions

$\mathcal{U}_2 = \{ f \in V \mid f(x) = -f(-x) \}$ ← odd functions

You can prove that U_1, U_2 are subspaces

Then $\mathcal{U}_1 + \mathcal{U}_2$ is a direct sum.

Pf. Recall that $\mathcal{U}_1 + \mathcal{U}_2$ is a direct sum if

$\mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$.

If $f \in V$ is both even and odd,

$$-f(-x) = f(x) = f(-x) \quad \forall x.$$

$$\Rightarrow f(x) = 0 \quad \forall x$$

$\square$

4.

I Linear independence

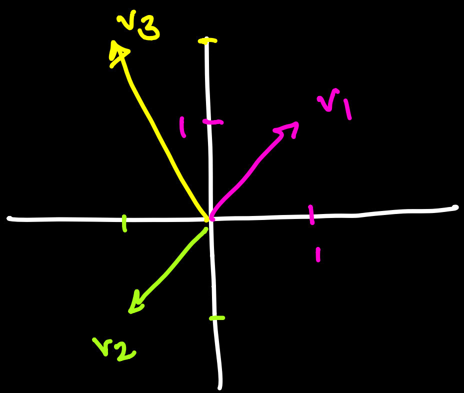
<u>Definition</u>: A list $v_1, v_2, \cdots \in V$ is <u>linearly independent</u> if $a_1 v_1 + \cdots + a_m v_m = 0 \iff a_1, \ldots, a_m = 0$ for any $m \in \mathbb{N}_{\geq 0}$ and $a_i \in F$

An empty list is also defined to be linearly independent.

check this if you feel it is a useful characterization

$\text{Span}(v_1) + \cdots + \text{Span}(v_m)$ is a direct sum $\iff v_1, \ldots, v_m$ are linearly independent.

<u>Ex</u>.



$v_1 = (1,1)$

$v_2 = (-1,-1)$

$v_3 = (2,-1)$

v_1, v_2 is <u>not</u> linearly independent

v_1, v_3 <u>is</u>.

$\text{Span}(v_1, v_2, v_3) = \mathbb{R}^2$

Pf. $(\Leftarrow)$ Let $\vec{0} = u_1 + \dots + u_m$ with $u_i \in \operatorname{Span}(v_i)$.

Then $\exists\, a \in F$ s.t. $u_i = a v_i$.

So $\vec{0} = a_1 v_1 + \dots + a_m v_m$.

By assumption, $a_i = 0 \ \forall i$.

So $u_i = 0 \cdot v_i = 0$.

By the Proposition from last lecture,

$$\operatorname{span}(v_1) + \dots + \operatorname{span}(v_m) \text{ is a direct sum.}$$

$(\Longrightarrow)$ Exercise.

Ex. $v_1 = (1,0,0)$
$v_2 = (0,1,0)$
$v_3 = (1,1,0)$
$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ this is not linearly independent.

$$v_3 = v_1 + v_2$$

$\operatorname{span}(v_1, v_2, v_3) =$ the xy-plane.

Def A list $v_1, \ldots, v_m$ is a basis for a vector space V if it is both spanning and linearly independent.

This seems quite useful

Ex Any 2 vectors in $\mathbb{R}^2$ that are not scalar multiples give a basis.

<u>Prop</u> (2.29) A list $v_1, \dots, v_n$ is a basis for V $\iff$ every $v \in V$ can be written uniquely in the form

$$(\star) \quad v = c_1 v_1 + \dots + c_n v_n, \quad \text{where } c_i \in F.$$

<u>Pf</u>. ($\Rightarrow$) First suppose that $v_1, \dots, v_n$ is a basis for V. Since $v_1, \dots, v_n$ span V, $\exists\ a_1, \dots, a_n$ s.t. $(\star)$ holds. By the previous Prop, $\mathrm{span}(v_1) + \dots + \mathrm{span}(v_m)$ is a direct sum, and so, by definition, $(\star)$ is the unique way to write v.

($\Leftarrow$) Suppose $(\star)$ holds $\forall\ v \in V$. Then $\mathrm{span}(v_1, \dots, v_m)$ is by definition V. Letting $\vec{v} = 0$, there is a unique way to write $\vec{0} = c_1 v_1 + \dots + c_m v_m$; we must therefore have $c_1 = \dots = c_m = 0$. $\square$

BUT CAN WE USE IT? DO BASES EXIST?

building block for a theorem.

Lemma

Any minimal spanning list is linearly independent.

Any maximal linearly independent list is spanning.

Here, a minimal spanning list is one in which you cannot remove an element and still span, a maximal linearly independent list is one in which you can't add an element and still be linearly independent.

Pf. First we show that a minimal spanning list is linearly independent. Suppose $v_1, v_2, \ldots$ is a minimal spanning list but is not linearly independent. Then $\exists$ N and $a_i \in F$ with some $a_i \neq 0$ s.t.

$$\sum_{i=1}^{N} a_i v_i = 0.$$

$\underbrace{\text{WLOG}}_{\text{without loss of generality}}$ assume $a_1 \neq 0$, so $-\sum_{i=2}^{N} a_i v_i = a_1 v_1$.

Then $v_1 = \sum_{i=2}^{N} -\frac{a_i}{a_1} v_i$.

So the list $v_2, v_3, \ldots$ is still spanning. This contradicts the assumption that $v_1, v_2, \ldots$ is minimal.

We now show that a maximally independent list is spanning.

If $v_1, v_2, \ldots$ is linearly independent and $u \notin \operatorname{span}(v_1, \ldots, v_m)$ then $v_1, \ldots, v_m, u$ is linearly independent. Else we could find $N \in \mathbb{N}$ $c_i, d \in F$ not all zero with $d \cdot u + \sum_{i=1}^{N} c_i v_i = 0$.

$u \notin \operatorname{span}(v_1, \ldots, v_m) \Rightarrow d = 0$, otherwise we could write $u = -\sum_{i=1}^{N} \frac{c_i}{d} v_i$.

So $\sum_{i=1}^{N} c_i v_i = 0$, contradicting linear independence.

Thus, $u \in \operatorname{span}(v_1, v_2, \ldots)$ and $v_1, v_2, \ldots$ is spanning

Prop If V is a vector space with a finite spanning list, V has a basis.

Pf Take a finite spanning list. If it is minimal, it is linearly independent, and therefore a basis. If not, delete an element, and repeat. This process terminates because we started with a finite list.