OBJECTIVE

Show that any two bases of a finite-dimensional vector space have the same size.

RECAP

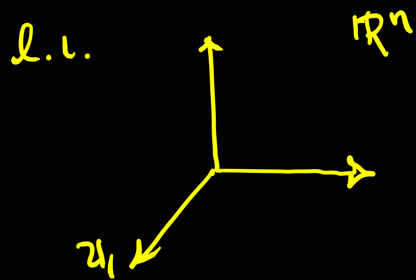
1. A list $v_1, \dots, v_m$ is spanning if every $v \in V$ can be written $v = a_1 v_1 + \dots + a_m v_m$, $a_i \in \mathbb{F}$.

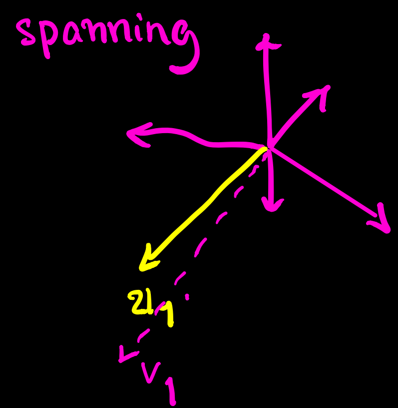
2. A list $v_1, \dots, v_m$ is linearly independent if
$$\vec{0} = a_1 v_1 + \dots + a_m v_m \implies a_1 = 0, \dots, a_m = 0.$$

3. A list $v_1, \dots, v_m$ is a basis if it is both spanning and linearly independent.

Lemma (Steinitz) Suppose that $u_1, \dots, u_m$ is linearly independent. Suppose that $v_1, \dots, v_n$ is spanning. Then $m \leq n$.

We can reorder $v_1, \dots, v_n$ so that for every $k \leq m$ the list $u_1, \dots, u_k, v_{k+1}, \dots, v_n$ is spanning.





Pf. The v_i are spanning, so we can write

(*) $u_1 = a_1v_1 + \dots + a_nv_n$, $a_i \in F$,

some $a_i \neq 0$.

WLOG, assume $a_1 \neq 0$.

Then $v_1 = a_1^{-1}u_1 - a_1^{-1}a_2v_2 - \dots - a_1^{-1}a_nv_n$. (*)

If $w = b_1v_1 + \dots + b_nv_n$, then substituting in (*) yields

$$w = b_1\left[a_1^{-1}u_1 - a_1^{-1}a_2v_2 - \dots - a_1^{-1}a_nv_n\right] + b_2v_2 + \dots + b_nv_n.$$

$$= b_1a_1^{-1}u_1 + (b_2 - b_1a_1^{-1}a_2)v_2 + \dots + (b_n - b_1a_1^{-1}a_n)v_n$$

Any element in $\operatorname{span}(v_1, \dots, v_n)$ can be written in terms of $u_1, v_2, \dots, v_n$.

$\Rightarrow$ $\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(u_1, v_2, \dots, v_n)$.

Iterate: Suppose we've shown that

$$\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(u_1, \dots, u_k, v_{k+1}, \dots, v_n).$$

Can we show that

$$\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(u_1, \dots, u_{k+1}, v_{k+2}, \dots, v_n)?$$

[Instead of showing

u_1 case $\Rightarrow$ u_2 case

u_2 $\Rightarrow$ u_3

u_3 $\Rightarrow$ u_4

$\vdots$]

We show that, for general k,

u_k case $\Rightarrow$ u_{k+1} case

Assume that $u_1, \dots, u_k, v_{k+1}, \dots, v_n$ span.

We can write

$$u_{k+1} = a_1 u_1 + \dots + a_k u_k + b_{k+1} v_{k+1} + \dots + b_n v_n.$$

I claim that $\exists$ some $b_i \neq 0$.

If not, I could write $u_{k+1} = a_1 u_1 + \dots + a_k u_k$,

contradicting linear independence.

WLOG, assume $b_{k+1} \neq 0$.

Rearranging,

$$v_{k+1} = b_{k+1}^{-1} u_{k+1} - b_{k+1}^{-1} a_1 u_1 - \dots - b_{k+1}^{-1} a_k u_k$$

$$- b_{k+1}^{-1} b_{k+2} v_2 - \dots - b_{k+1}^{-1} b_n v_n .$$

$\Rightarrow$ If I can write w as a linear combination of

$u_1, \dots, u_k, v_{k+1}, \dots, v_n$, I can write w as a

linear combination of $u_1, \dots, u_{k+1}, v_{k+2}, \dots, v_n$.

We conclude that $\operatorname{span}(u_1, \dots, u_k, v_{k+1}, \dots, v_n)$

$$\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(u_1, \dots, u_{k+1}, v_{k+2}, \dots, v_n).$$

Eventually, $k=m$, and we arrive at a list

$$u_1, \dots, u_m, v_{m+1}, \dots, v_n.$$

that is spanning.

Because we can fit $u_1, \dots, u_m$ into $v_1, \dots, v_n$

$$m \le n.$$

$\square$

WHAT WAS OUR METHOD OF PROOF ?

INDUCTION

A method of proof used to show that a statement holds for all natural numbers.

1. Establish a base case: show that your statement holds for 0.

   → or whatever # you start counting at.

2. Inductive step: Assume that the statement holds for k, and prove it for $k+1$.

"you can climb as high as you would like on a ladder".

Corollary Any two bases of a finite dimensional vector space have the same length.

something that follows "immediately" or "almost immediately" from another statement

Proof If B_1 and B_2 are both bases then B_1 is linearly independent and B_2 is spanning. By the Lemma, the length of $B_1 \leq$ the length of B_2.

$$|B_1| \leq |B_2|$$

Reversing the roles of B_1, B_2

$$|B_2| \leq |B_1|.$$

$$|B_1| = |B_2|$$

$\square$

Def The dimension of a vector space is the length of a basis.

Ex. $\dim(\mathbb{R}^n) = n.$

A basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(\mathcal{P}_n(\mathbb{R})) = n+1$

A basis: $\{1, x, x^2, \ldots, x^n\}$

There is a $\underbrace{\text{bijection}}_{\substack{f(x)=f(y) \Rightarrow x=y \\ \exists x \text{ s.t. } f(x)=z \ \forall z \in V}}$ from $F^{\dim(V)} \xrightarrow{f} V$.

If $(v_1, \ldots, v_n)$ a basis for V.

$$f : \begin{array}{c} \\ \\ \\ \text{i^{th} component} \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow v_i$$

Define $f\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 f\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 f\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + a_n f\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

"Extending linearly"

$[0] \in \mathbb{R}$ vs. $\begin{bmatrix}0\\0\end{bmatrix} \in \mathbb{R}^2$

$1 \longleftrightarrow \begin{bmatrix}1\\0\end{bmatrix}$