MATH 113
Linear Algebra
Fall 2020

Challenge Problems Week 2

Just for fun. You won't be tested on this and you won't get points for it. Feel free to turn them in with your homework or email me or come chat with me about them.

1. Let $V = \mathbb{C}^n$ be a vector space over $F = \mathbb{C}$. Let $\{v_1, \dots, v_n\}$ and $\{u_1, \dots, u_n\}$ be two bases for V. Prove that there are continuous functions $x_i : [0, 1] \to \mathbb{C}^n$ with $x_i(0) = v_i$ and $x_i(1) = u_i$, and such that $\{x_1(t), \dots, x_n(t)\}$ is a basis for every t. Is this still true if $V = \mathbb{R}^n$ and $F = \mathbb{R}$?

2. Here is an old concept from machine learning that characterizes linearly independent vectors in $\mathbb{R}^n$. Stripped of its machine-learning trappings, the concept is purely a mathematical understanding of geometry. Let $\{v_1, \dots, v_m\}$ be a list of vectors.

Let $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$ be some list of k distinct integers between 1 and m. Define a function

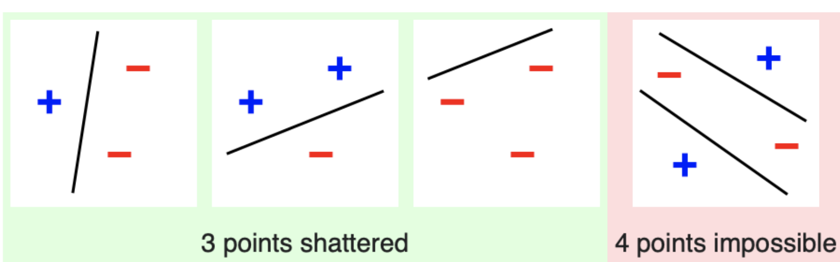
$$f_{i_1,\dots,i_k} : \{v_1, \dots, v_m\} \to \{0, 1\}$$

by

$$f_{i_1,\dots,i_k}(v_j) = 1 \qquad \text{if } j \in \{i_1, \dots, i_k\}$$

and $f_{i_1,\dots,i_k}(v_j) = 0$ otherwise. Thus, each function $f_{i_1,\dots,i_k}$ is *labeling* the vectors with either 0 or 1. For some labeling, it may be possible to separate the data so that all of the "0" labeled data lies on one side of a *hypersurface* (an $n - 1$ dimensional subspace that is possibly translated away from the origin) and all of the "1" labeled data lies on the other side. If this separation is possible for *every* labeling, we say that the the list $\{v_1, \dots, v_m\}$ is *shattered.*

For example, if we are in $\mathbb{R}^2$, then three points can be shattered but four cannot:



This suggests the following theorem.

THEOREM 0.1. *Assume without loss of generality that v_1 is the zero vector. The vectors $\{v_1, \dots, v_m\}$ are shattered if and only if the list $\{v_2, \dots, v_m\}$ is linearly independent.*

Can you see it?? Can you prove it??? Warning: this is really quite challenging. The proof is much longer and more complex than anything we've seen, and it really involves diving deep into the vector geometry. Think about it if it's fun for you, but stop thinking about it as soon as it's not fun. Hint: Assuming they're linearly independent and showing they can be shattered is a bit easier than the other direction: consider the *convex hulls* of the two different sets of labeled points. A convex hull of a set of vectors $\{u_1, \ldots, u_\ell\}$ is the set of linear combinations $\{t_1u_1 + t_2u_2 + \cdots + t_\ell u_\ell\}$, where $t_i \geq 0$ for all i and $t_1 + t_2 + \cdots + t_\ell = 1$. If the sets can't be shattered, what does that mean geometrically about their convex hulls? What does this imply about linear independence?

You can also read the proof here on page 37. It needs a Lemma to be complete; the Lemma is stated and proved on page 36.

The maximum number of vectors that can be shattered is called the *VC dimension* or Vapnik-Chervonenkis dimension, and is a measure of how well a group of functions can be learned by a binary classification algorithm. It is an "old" concept in this rapidly-changing field, and its usefulness is apparently hotly debated (see the discussion here).