MATH 113
LINEAR ALGEBRA
FALL 2020

Challenge Problems Week 1

Just for fun. You won't be tested on this and you won't get points for it. Feel free to turn them in with your homework or chat with me about them.

1. Let p be a prime, and consider the field $\mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$ with addition and multiplication modulo p, as discussed in lectures. We will prove the existence of multiplicative inverses $1/a = a^{-1}$ for $a \neq 0$. So for this question, you may not quote this fact.

(a) Let $a \in \mathbb{F}_p$, $a \neq 0$, be given. Show that for any $x, y \in \mathbb{F}_p$, if $ax = ay$ (using $\mathbb{F}_p$ multiplication) then $x = y$. [Hint: you will have to use the facts that $a \neq 0$ and that p is prime.]

(b) By considering the set $\{a0, a1, \ldots, a(p-1)\}$, or otherwise, deduce that there exists $b \in \mathbb{F}_p$ such that $ab = 1$ (over $\mathbb{F}_p$).

(c) Determine when the set $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$ with addition and multiplication modulo n is a field..

2. We saw that a vector space is defined by 8 Fundamental Properties, and other desirable properties (like uniqueness of additive inverses) follows from these 8. This problem asks how necessary all 8 properties are.

(a) Recall that Property 4 states that there for each $\vec{x} \in V$ is an additive inverse $-\vec{x}$ such that $\vec{x} + (-\vec{x}) = 0$. Suppose we replace Property 4 with the condition that for each $\vec{v} \in V$ the element $(-1)\vec{v}$ is its additive inverse. Show that, with this replacement, Property 8: $1 \cdot \vec{v} = \vec{v}$ for all $\vec{v} \in V$ can be deduced from the other 7 properties.

(b) Show by example that without this replacement Property 8 cannot be deduced from the remaining 7 properties.

(c) In general, can any of the 8 properties be deduced from the other 7? Can any two of the 8 properties be deduced from the other 6? How good can we get?