Last time we ended with orthogonal complement $S^\perp$ and proved some stuff.

Prop. If V is a finite dimensional inner product space and $\mathcal{U} \subseteq V$ is a subspace then

(1) $V = \mathcal{U} \oplus \mathcal{U}^\perp$

$$\begin{array}{ccccc} \text{(2)} & v & = & u & + & w \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \end{array}$$

(2) $v = u + w$ and if $\{u_1, \dots, u_k\}$ is an orthonormal basis for $\mathcal{U}$ then

$$u = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_k \rangle u_k .$$

Pf. $v \in V$.

Set $u = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_k \rangle u_k \in \mathcal{U}$.

$w = v - \mathcal{U}.$

Is $w \in \mathcal{U}^\perp$?

$\langle w, u_i \rangle = \langle v, u_i \rangle - \langle u, u_i \rangle$

$= \langle v, u_i \rangle - \langle v, u_i \rangle$

inner product "selects for" the i^{th} element.

$= 0.$

$\Rightarrow w \in \{u_1, \dots, u_k\}^\perp.$

Prop: $S^\perp = \text{span}(S)^\perp.$

$\Rightarrow w \in \text{span}(u_1, \dots, u_k)^\perp = \mathcal{U}^\perp.$

$\Rightarrow V = \mathcal{U} + \mathcal{U}^\perp.$

Prop: $\mathcal{U} \cap \mathcal{U}^\perp = \{0\}.$

$\Rightarrow V = \mathcal{U} \oplus \mathcal{U}^\perp.$

$\Rightarrow$ The only way to write v as something in $\mathcal{U}$ plus something in $\mathcal{U}^\perp$ is

$$v = \underbrace{\left(\langle v, u_1 \rangle u_1 + \cdots + \langle v, u_k \rangle u_k \right)}_{u} + (v - u).$$

$\square$

Def. The <u>orthogonal projection map</u> from V onto a subspace $\mathcal{U}$ is the map

$$V = \mathcal{U} \oplus \mathcal{U}^\perp \longrightarrow V$$
$$u + w \longmapsto u$$

Note: well-defined by the Proposition

Cor. (i) $\dim(\mathcal{U}) + \dim(\mathcal{U}^\perp) = \dim(\mathcal{V})$

(ii) $(\mathcal{U}^\perp)^\perp = \mathcal{U}$

$\Big\}$ $\mathcal{V}$ finite dimensional

Pf. (i) $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$.

We showed if $\{u_1, \ldots, u_k\}$ is a basis for $\mathcal{U}$ and $\{w_1, \ldots, w_\ell\}$ is a basis for $\mathcal{U}^\perp$ then $\{u_1, \ldots, u_k, w_1, \ldots, w_\ell\}$ is a basis for $\mathcal{V}$.

$$\dim(\mathcal{V}) = k + \ell = \dim(\mathcal{U}) + \dim(\mathcal{U}^\perp).$$

(ii) $u \in \mathcal{U}$ and $w \in \mathcal{U}^\perp$ then $\langle u, w \rangle = 0$.

$\rightsquigarrow \mathcal{U} \subseteq (\mathcal{U}^\perp)^\perp$.

By (i)
$$\left.\begin{aligned} \dim(\mathcal{U}) + \dim(\mathcal{U}^\perp) &= \dim(\mathcal{V}) \\ \dim(\mathcal{U}^\perp) + \dim((\mathcal{U}^\perp)^\perp) &= \dim(\mathcal{V}) \end{aligned}\right\} \dim(\mathcal{U}) = \dim((\mathcal{U}^\perp)^\perp)$$

$\rightsquigarrow \quad \mathcal{U} = (\mathcal{U}^{\perp})^{\perp}.$ $\square$

"cute application" of inner products

We saw on HW that if $\{v_1, \dots, v_n\}$ is a basis of V it induces a dual basis $\{f_1, \dots, f_n\}$ of V^* defined by $f_i(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

If V has an inner product $\langle , \rangle$ we can define $g_1, \dots, g_n \in V^*$ by $g_i(v) = \langle v, v_i \rangle$.

When does $\langle , \rangle$ produce $g_1, \dots, g_n$ where $g_i = f_i$?

Precisely when $v_1, \dots, v_n$ is an orthonormal basis w.r.t. $\langle \, , \, \rangle$.

To show: $g_i(v_j) = \langle v_j, v_i \rangle \overset{\text{if orthonormal}}{=} \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases} = f_i(v_j)$

<u>Cor</u> (Riesz Representation Thm) V is a finite-dim. inner product space and $\varphi \in V^*$ then $\underbrace{\exists!}_{\text{"there exists a unique"}}$ $u \in V$ s.t. $\varphi(v) = \langle v, u \rangle \quad \forall \; v \in V$.

<u>Pf</u>. Can write $\varphi = a_1 g_1 + \dots + a_n g_n$, $a_i \in F$

$g_i(v) = \langle v, v_i \rangle$.

$$\varphi(v) = a_1 g_1(v) + \dots + a_n g_n(v)$$

$$= a_1 \langle v, v_1 \rangle + \dots + a_n \langle v, v_n \rangle$$

$$= \langle v, \underbrace{\bar{a}_1 v_1 + \dots + \bar{a}_n v_n}_{u.} \rangle .$$

uniqueness: If $\varphi(v) = \langle v, u \rangle = \langle v, u' \rangle \quad \forall \ v$

then $0 = \langle v, u \rangle - \langle v, u' \rangle$

$= \langle v, u - u' \rangle .$

Trick: Pick $v = u - u'$.

$$0 = \langle u - u', u - u' \rangle = \| u - u' \|^2 .$$

$\Rightarrow \quad u - u' = 0 \quad$ or $\quad u = u'$.

$\square$

" Every linear functional can be represented by a vector in V."