Recall: $\boxed{\dim(V) = \dim(W)} \iff \exists$ an isomorphism

$$T : V \to W. \quad (\star)$$

Cor. If $\{v_1, \ldots, v_n\}$ is a basis for V and $T : V \to W$ is an isomorphism, $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

Pf. Spanning: let $w \in W$. T an isomorphism, $\exists\ v \in V$ with $T(v) = w$.

Write $v = a_1 v_1 + \ldots + a_n v_n$.

Then
$$\begin{aligned} w &= T(a_1 v_1 + \ldots + a_n v_n) \\ &= a_1 T(v_1) + \ldots + a_n T(v_n). \end{aligned}$$

$\Rightarrow \{T(v_1), \ldots, T(v_n)\}$ span.

Linear independence: $\dim(W) = \dim(V) = n$.

$\Rightarrow$ A spanning list of size n is a basis for W.

$\Rightarrow$ $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

COR → bases

$\Leftarrow$ result about dim / iso

$\Leftarrow$ Rank-Nullity

$\Leftarrow$ Exploited properties of bases

How did we prove $(\star)$?

We fixed a basis $\{v_1, \dots, v_n\}$ for V

$\{w_1, \dots, w_n\}$ for W

and we constructed a map $T : V \to W$

$v_i \longmapsto w_i$.

What about the map $S : W \to V$?

$w_i \longmapsto v_i$

$$S \circ T : V \xrightarrow{T} W \xrightarrow{S} V$$

$$v_i \longmapsto w_i \longmapsto v_i.$$

$S \circ T$ is just the identity map on V!

(written I_V).

$$T \circ S : W \xrightarrow{S} V \xrightarrow{T} W$$

$$w_i \longmapsto v_i \longmapsto w_i .$$

$T \circ S$ is just the identity map on W!

(written I_W).

We call S the <u>inverse</u> of T, and write $S = T^{-1}$.

We call T <u>invertible</u> if T has an inverse:

if $\exists$ a map $T^{-1} : W \to V$ s.t.

$$T \circ T^{-1} = I_W \quad \text{and} \quad T^{-1} \circ T = I_V .$$

<u>Prop</u>. Inverses are unique.

Pf. If S and S' are two inverses of $T: V \to W$

$$S' = S' \circ I_W \underset{\substack{\uparrow \\ \text{by defn of} \\ \text{inverse}}}{=} S' \circ (T \circ S) = (S' \circ T) \circ S$$

$$= I_V \circ S$$

$$= S. \qquad \square$$

Note. This justifies our notation T^{-1}!

Examples

① $T: F^n \longrightarrow F^n$

$v \longmapsto a \cdot v$, $\quad \underbrace{a \in F \text{ is non-zero}}_{\substack{a \in F^* \\ \rotatebox{90}{$\sim$} \\ F \setminus \{0\}}}$.

$T^{-1}: F^n \longrightarrow F^n$

$v \longmapsto a^{-1} \cdot v$.

$$\begin{array}{ccccc} F^n & \xrightarrow{T} & F^n & \xrightarrow{T^{-1}} & F^n \\ v & \longmapsto & a \cdot v & \longmapsto & \underbrace{a^{-1} \cdot (a}_{1} \cdot v) = v. \end{array}$$

$$\begin{array}{ccccc} F^n & \xrightarrow{T^{-1}} & F^n & \xrightarrow{T} & F^n \\ v & \longmapsto & a^{-1} \cdot v & \longmapsto & \underbrace{a \cdot (a^{-1}}_{1} \cdot v) = v. \end{array}$$

② $T : P_n(\mathbb{R}) \longrightarrow \mathbb{R}^n$

$$x^i \longmapsto \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ slot}.$$

$T^{-1} : \mathbb{R}^n \longrightarrow P_n(\mathbb{R})$

$$i^{th} \rightarrow \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \longmapsto x^i$$

③ $T : P_n(\mathbb{R}) \longrightarrow \mathbb{R}^n$

$$1 + x + \dots + x^i \longmapsto \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ slot}$$

$T^{-1} : \mathbb{R}^n \longrightarrow P_n(\mathbb{R})$

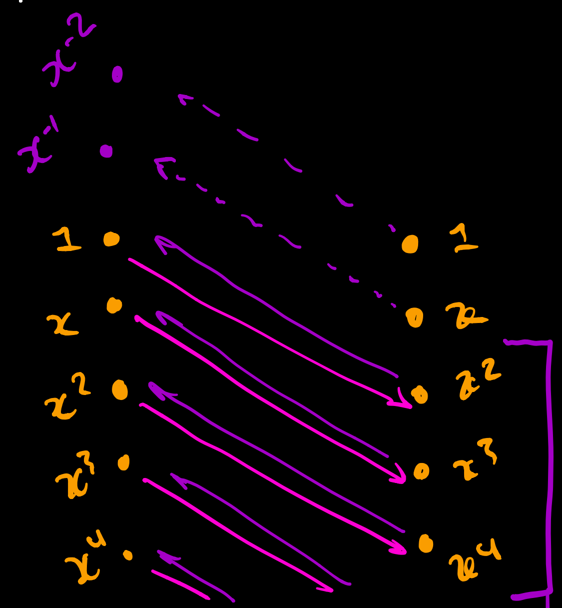
$$i^{th} \rightarrow \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \longmapsto 1 + x + \dots + x^i .$$

<u>Non-ex.</u>

① $T : P(\mathbb{R}) \longrightarrow P(\mathbb{R})$

$$p(x) \longmapsto x^2 p(x)$$

is <u>not</u> invertible.



② $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Then $T \circ S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

$\rightsquigarrow \quad T \circ S = I_{\mathbb{R}^2}$.

However, $S \circ T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

$\rightsquigarrow \quad S \circ T \neq I_{\mathbb{R}^3}$.

"We call S a 'right inverse' of T."

<u>Thm</u>. A linear map is $\boxed{\text{invertible}}$ $\Longleftrightarrow$ it is an isomorphism.

<u>Pf</u>. ($\Longrightarrow$) Assume $T: V \to W$ is invertible.

Let $T^{-1}: W \to V$ be the inverse of T.

If $\underbrace{T(v) = 0}$, then
$$
\begin{aligned}
v &= I_V(v) \\
&= (T^{-1} \circ T)(v) \\
&= T^{-1}(0) \\
&= 0.
\end{aligned}
$$

$\Rightarrow$ $\operatorname{null}(T) = \{0\}$.

$w \in W$. $w = I_W(w) = (T \circ T^{-1})(w) = T(T^{-1}(w))$.

Setting $v = T^{-1}(w)$, $T(v) = w$.

$\Rightarrow$ T is surjective.

($\Leftarrow$) Assume T is isomorphic.

By Corollary, if $\{v_1, \dots, v_n\}$ is a basis for V, $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.

$$\{\underset{w_1}{\underset{\parallel}{T(v_1)}}, \dots, \underset{w_n}{\underset{\parallel}{T(v_n)}}\}$$

Define: $S : W \to V$, $w_i \mapsto v_i$ and extending linearly.

$S \in \mathcal{L}(W, V)$ by construction.

$$S \circ T(a_1 v_1 + \dots + a_n v_n) =$$

$$a_1 S \circ T(v_1) + \dots + a_n S \circ T(v_n) =$$

$$a_1 S(w_1) + \dots + a_n S(w_n) =$$

$$a_1 v_1 + \dots + a_n v_n.$$

Also need to check: $T \circ S(w) = w$. $\overset{?}{\square}$

Textbook Defines isomorphism to be an invertible map.