Objective

① State the 'rank-nullity' theorem

$T: V \to W$ linear map

We call the nullspace of T (also called the kernel)

$$\mathrm{null}(T) = \{ v \in V \mid T(v) = 0 \}$$

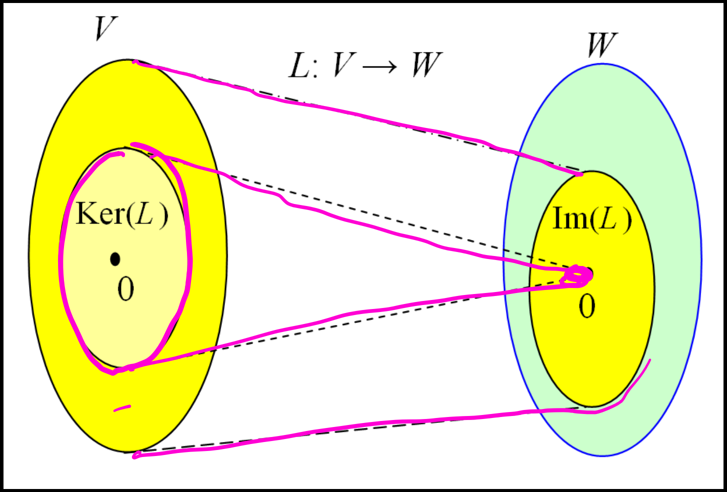
We call the image of T (range)

$$\mathrm{im}(T) = \{ w \in W \mid \exists\, v \in V \text{ with } T(v) = w \}$$

Ex $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$(x, y) \mapsto (x, 0, 0)$$

$\mathrm{null}(T) = \{ (0, y) \mid y \in \mathbb{R} \}$ $\quad$ $\mathrm{im}(T) = \{ (x, 0, 0) \mid x \in \mathbb{R} \}$



V
W
L: V → W
Ker(L)
0
Im(L)
0

Ex $P(\mathbb{R}) \longrightarrow P(\mathbb{R})$ is $\underbrace{\text{injective}}$ but not surjective.

$p \overset{T}{\longmapsto} x^2 p$

$T(v) = T(w) \Rightarrow v = w$.

$\text{null}(T) = \{p \mid T(p) = 0\}$

$= \{p \mid \underbrace{x^2 \cdot p} = 0\}$

$= \{p = 0\}$

representing the function $z : x \mapsto 0$.

$\forall x \in \mathbb{R}, \exists (u,v) \in \mathbb{R}^2, T(u,v) = x$.

$\mathbb{R}^2 \longrightarrow \mathbb{R}$ is $\overbrace{\text{surjective}}$ but not injective.

$(x,y) \overset{T}{\longmapsto} x$

$\text{null}(T) = \{(x,y) \mid T(x,y) = 0\}$

$= \{(x,y) \mid x = 0\} = \{(0,y)\}$.

$F^n \longrightarrow F^n$ is an isomorphism.

$v \overset{T}{\longmapsto} a \cdot v, \quad a \neq 0$

$\text{null}(T) = \{v \mid T(v) = 0\} = \{v \mid a \cdot v = 0\} = \{v = 0\}$.

Prop Injectivity is equivalent to $\text{null}(T) = \{\vec{0}\}$.
$T: V \to W$

Pf. $\boxed{(\Longrightarrow)}$ Injectivity implies $\text{null}(T) = \{\vec{0}\}$.

Let $v \in V$. suppose $T(v) = 0$.

Because T is a linear transformation, $T(0) = 0$.

[So $\{\vec{0}\} \subseteq \text{null}(T)$]

So $T(v) = T(0)$

By injectivity, $v = 0$.

$\boxed{(\Longleftarrow)}$ Assume $\text{null}(T) = \{\vec{0}\}$. Suppose $u, v \in V$ with $T(u) = T(v)$.

Rearranging, $T(u) - T(v) = 0$.

By linearity, $T(u-v) = 0$. $\Longrightarrow$ $u - v = 0$, or $u = v$.

$\square$

Different way to prove $\boxed{\Longleftarrow}$

Suppose T is not injective.

Then $\exists\, v, u \in V$ with $v \neq u$ and $T(v) = T(u)$.

Rearranging, $T(v) - T(u) = 0$

$$T(v-u) = 0.$$

Since $v \neq u$, $v - u \neq 0$ and so $\{0\} \neq \text{null}(T)$

$\square$

PROOF VIA CONTRAPOSITIVE. $A \Rightarrow B$

$\text{not}(B) \Rightarrow \text{not}(A)$

If $\text{null}(T) = \{0\}$ BUT T is not injective, then by the work I've just done, $\text{null}(T) \neq \{0\}$

$\leadsto$ contradiction. T must be injective.

Injectivity of $T \iff \text{null}(T) = \{0\}$

Surjectivity of $T \iff \text{image}(T) = W$ (why?)

<u>Prop</u> null(T) is a subspace of V and image(T) is a subspace of W. → for you.

<u>Pf.</u> $\text{null}(T) \subseteq V$:

$0 \in \text{null}(T)$, since we saw previously that $T(0) = 0$.

If $v, u \in \text{null}(T)$ then $T(v+u) = T(v) + T(u)$

$$= 0 + 0 = 0$$

If $v \in \text{null}(T)$ then $T(a \cdot v) = a \cdot T(v) \quad \forall a \in F$

$$= a \cdot 0$$

$$= 0$$

<u>THM</u> [Rank-nullity theorem or fundamental theorem]

$$T \in \mathcal{L}(V, W)$$

$$\dim V = \dim \operatorname{null}(T) + \dim \operatorname{im}(T)$$

Pf Choose a basis $\{v_1, \ldots, v_n\}$ for null(T).

And a basis $\{w_1, \ldots, w_m\}$ for im(T).

Choose $\{u_1, \ldots, u_m\}$ where $T(u_i) = w_i$.

We want to show that $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ is a basis for V.

[$n+m = \dim(V)$ and $n+m = \dim \text{null}(T) + \dim \text{im}(T)$]

We need to show linear independence and spanning

① Linear independence.

Suppose $\exists\, a_1, \ldots, a_n \in F$ and $b_1, \ldots, b_m \in F$ such that
$$\sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j u_j = 0. \quad (*)$$

Apply T: $T\left[\sum a_i v_i + \sum b_j u_j\right] = T(0) = 0$

$$\sum a_i \cancel{T(v_i)}^{0} + \sum b_j \underbrace{T(u_j)}_{w_j} = 0$$

$$\sum b_j w_j = 0.$$

$\{w_1, \ldots, w_m\}$ basis $\Rightarrow b_j = 0 \;\; \forall j$.

(*) $\displaystyle\sum_{i=1}^{n} a_i v_i = 0$

$\{v_1, \ldots, v_n\}$ basis $\Rightarrow a_i = 0 \;\; \forall i$.

So all coefficients of (*) must be 0, and my list $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ is linearly independent.

② Spanning. Choose $x \in V$

I can write $T(x) = \sum_{i=1}^{m} b_i w_i$ for some $b_i \in F$.

$$= \sum_{i=1}^{m} b_i T(u_i)$$

$$T(x) - T\left(\sum b_i u_i\right) = 0$$

$$T\left[x - \sum b_i u_i\right] = 0.$$

So $x - \sum b_i u_i \in \text{null}(T)$

So I can write $x - \sum_{i=1}^{m} b_i u_i = \sum_{i=1}^{n} a_i v_i$, for some $a_i \in F$

Rearranging, $x = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{m} b_i u_i$.

$\rightsquigarrow \{v_1, \dots, v_n, u_1, \dots, u_m\}$ is spanning.

Since $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ is spanning and linearly independent, it is a basis for V.

$\square$