Thm. If $\mathbb{F} = \mathbb{C}$ and V is a finite-dimensional vector space, then any $T \in \mathcal{L}(V)$ has an eigenvalue.

Idea. By the Prop, all we need to do is find some $\lambda \in \mathbb{C}$ for which the map $T - \lambda \cdot I_V$ is not injective.

If I have a bunch of candidate eigenvalues $\lambda_1, \dots, \lambda_k$ I can test them all at once by checking if the composition

$$(T - \lambda_1 I_V) \circ \underbrace{(T - \lambda_2 I_V) \circ \dots \circ}_{\text{this looks like a factored polynomial.}} (T - \lambda_k I_V)$$

is injective.

If this sends some non-zero $v \in V$ to $\vec{0}$, then at least one of the $T - \lambda_i I_V$ s is not injective.

Recall in the proof:

$$0 = \underbrace{\boxed{(a_0 + a_1 T + a_2 T^2 + \dots + a_n T^n)}}_{p(T)}(v)$$

$$= c(T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_n I)(v).$$

Need to check that "polynomials over operators" behave the same way as the usual polynomials.

We know that there is a map

phi

$$\mathcal{P}(F) \xrightarrow{\Phi} \{\text{polynomials in } T\}$$

$$\sum c_i z^i \longmapsto \sum c_i T^i.$$

Does this map respect multiplication? AKA

Does $\Phi(p(z) \cdot q(z)) = \Phi(p(z)) \circ \Phi(q(z))$?

Thm. If V is finite-dimensional and $T \in \mathcal{L}(V)$ then the following are equivalent:

(1) T is diagonalizable

(2) V has a basis consisting of eigenvectors of T.

(3) There exists 1-dim. invariant subspaces $U_1, \ldots, U_n$ s.t. $V = U_1 \oplus \ldots \oplus U_n$.

(4) $V = E(\lambda_1) \oplus \ldots \oplus E(\lambda_k)$, where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues.

(5) $\dim(V) = \dim(E(\lambda_1)) + \ldots + \dim(E(\lambda_k))$.

Pf. (1) $\Leftrightarrow$ (2).

If T is diagonalizable, then there is some basis $\{v_1, \dots, v_n\}$ of V such that the matrix of T w.r.t. this basis is

$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & \cdots & v_n \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \end{array}$$

(by definition).

$T(v_i) = \lambda_i \cdot v_i$.

So v_i is an eigenvector for T.

Conversely, if $\{v_1, \dots, v_n\}$ is a basis of eigenvectors, then $\exists$ scalars $\lambda_1, \dots, \lambda_n$ s.t. $T(v_i) = \lambda_i v_i$.

The matrix of T w.r.t. this basis is $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

This is a diagonal matrix, so T is diagonalizable.

Pf. (5) $\Rightarrow$ (2).

For each $E(\lambda_i)$ choose a basis $\{v_1^i, v_2^i, \dots, v_{n_i}^i\}$.

The list of all basis elements of all $E(\lambda_i)$ has size equal to n by assumption.

$$\left[\dim(E(\lambda_1)) + \dots + \dim(E(\lambda_k)) = \#\{v_1^1, \dots, v_{n_1}^1\} + \dots + \#\{v_1^k, \dots, v_{n_k}^k\} = n = \dim(V)\right]$$

So $\{v_1^1, \dots, v_{n_1}^1, \dots, v_1^k, \dots, v_{n_k}^k\}$ is a basis if it is a linearly independent list.

$\boxed{v_1^1, v_2^1, \dots, v_{n_1}^1}$

When is $a_1 v_{i_1}^1 + a_2 v_{i_2}^2 + \dots + a_k v_{i_k}^k = 0$?

$$u_1 + u_2 + \dots + u_k = 0 \,?$$

$E(\lambda_1) \quad E(\lambda_2) \qquad E(\lambda_k)$

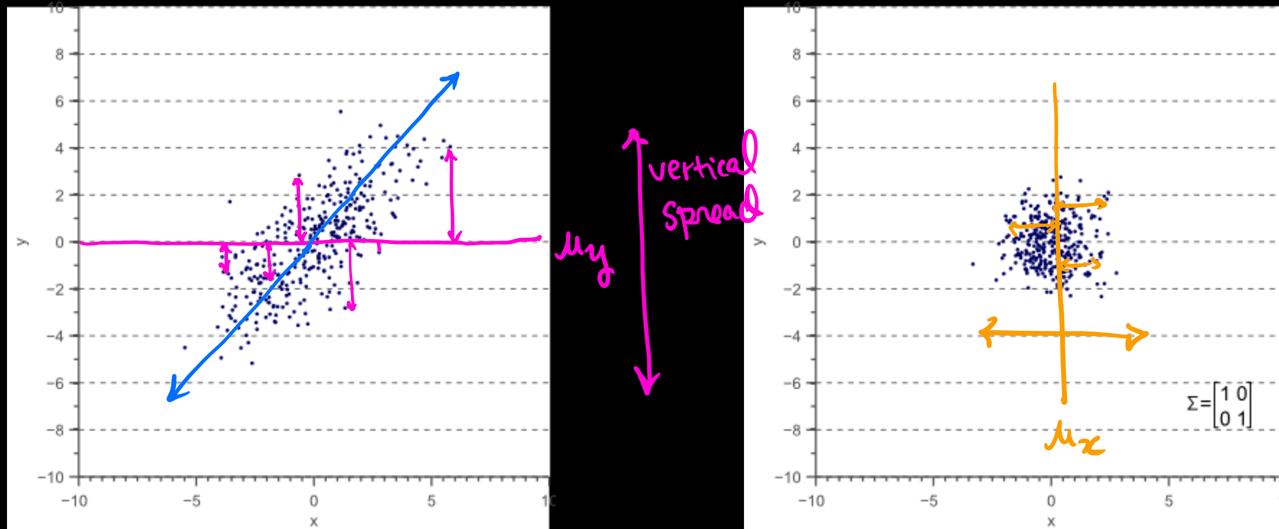
Each u_i is an eigenvector belonging to a distinct eigenvalue.

$\Rightarrow u_1 = u_2 = \dots = u_k = 0.$

$\Rightarrow a_1 = a_2 = \dots = a_k = 0.$

$\square$

Application: The shape of data. [Ref: computer vision for dummies]

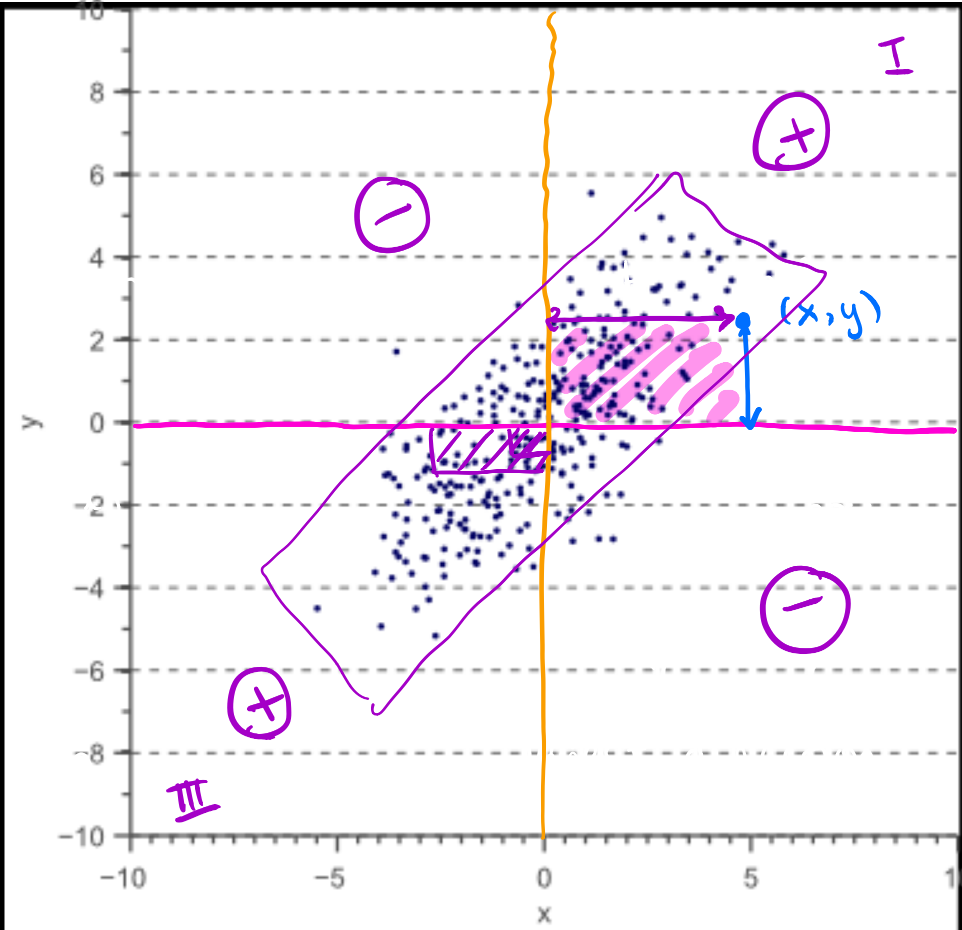


Say I have data points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$.

Want to understand: how spread out is my data?

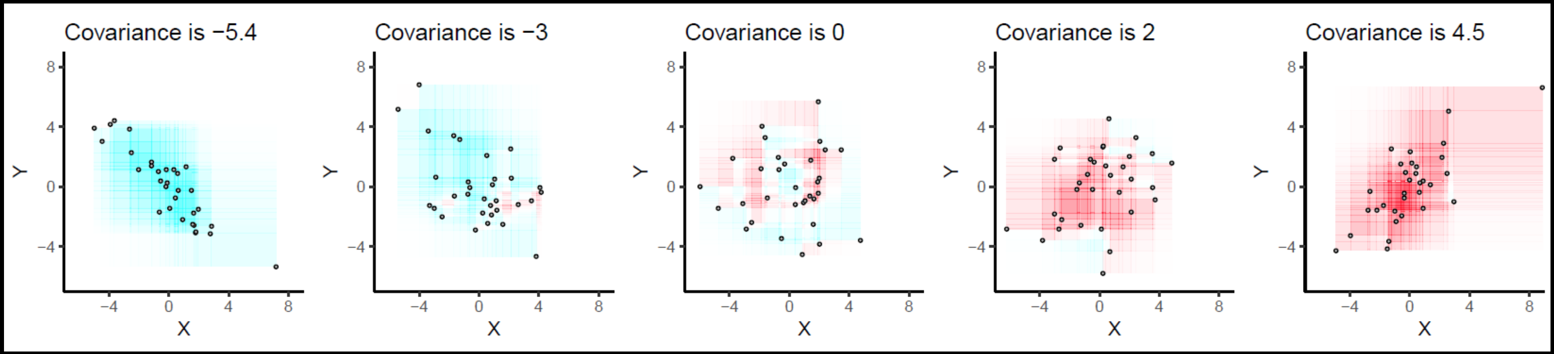
What is the standard deviation?

$$\sigma(y,y) = \frac{1}{N-1}\sum_{i=1}^{N}(y_i-\mu_y)^2 \qquad \sigma(x,x) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\mu_x)^2.$$



Covariance

$$\sigma(x,y) = \frac{1}{N-1}\sum_{i=1}^{N} (x_i-\mu_x)\cdot(y_i-\mu_y)$$

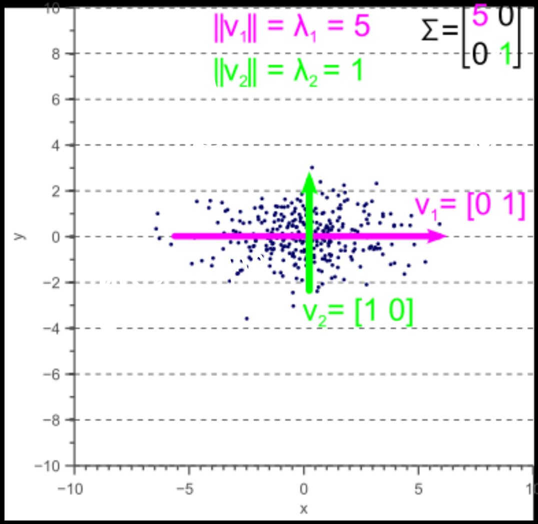

Covariance is −5.4
Covariance is −3
Covariance is 0
Covariance is 2
Covariance is 4.5
Y
X
8
4
0
−4
−4
0
4
8

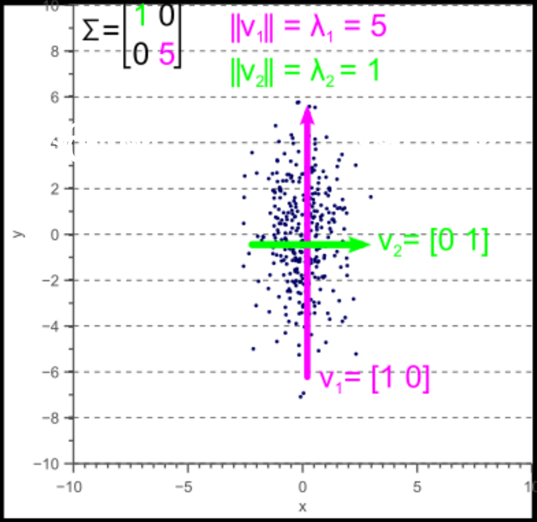
We can organize all "spread" info into a matrix, called the <u>covariance matrix</u>.

$$\Sigma = \begin{bmatrix} \sigma(x,x) & \sigma(y,x) \\ \sigma(x,y) & \sigma(y,y) \end{bmatrix}$$

capital sigma

If $\sigma(x,y) = \sigma(y,x) = 0$, Σ has a very nice interpretation:

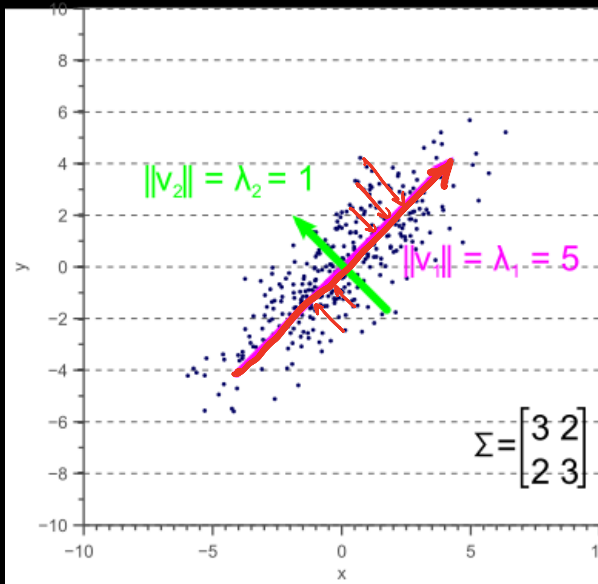


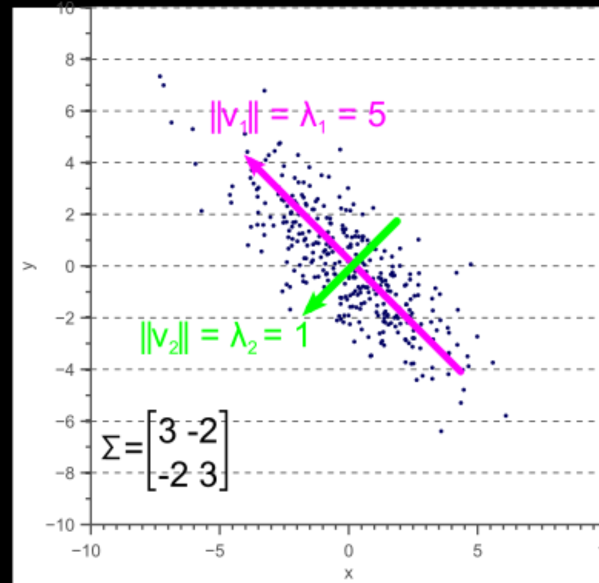


in this case, Σ is a diagonal matrix.

CRAZY FACT: the eigenvectors of Σ point in the directions of maximal spread!

Eigenvalues tell you by how much.



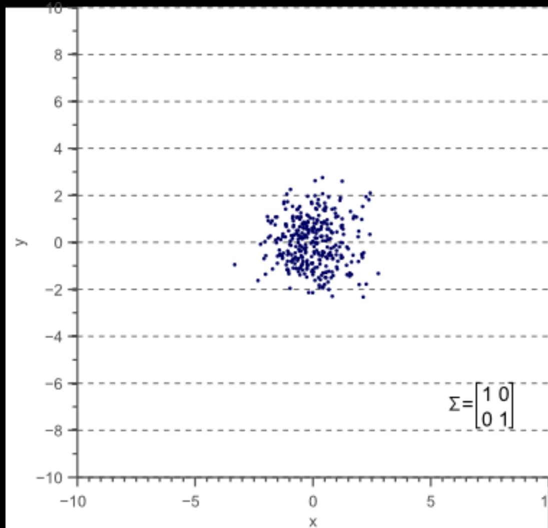


Principle component analysis.

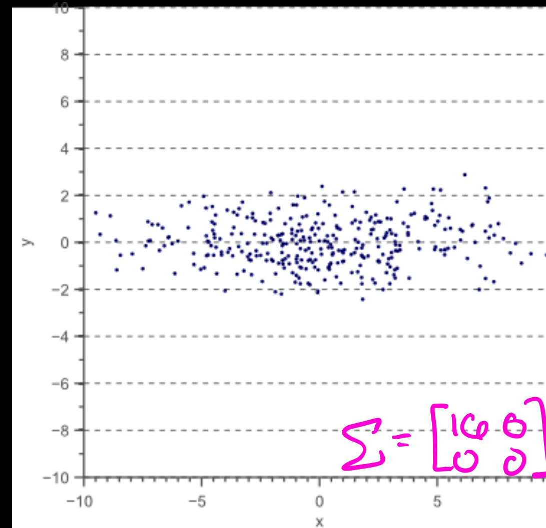
Crucial fact: Σ is <u>always</u> diagonalizable.

You can always find the principle components.

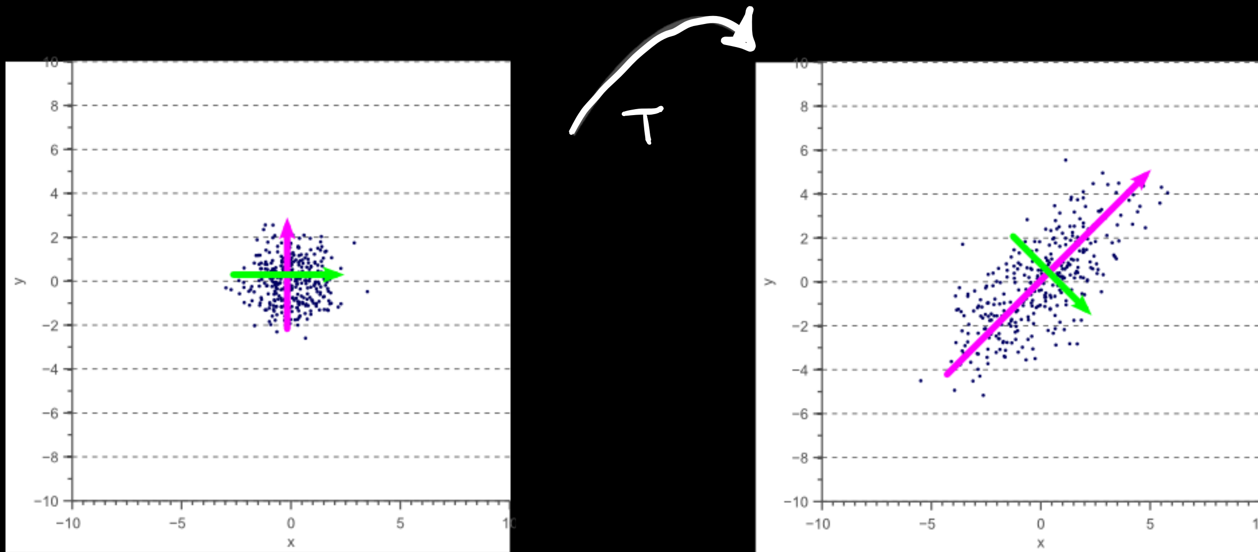
There always exists "white data", data for which Σ = identity, and a linear transformation T



$$\xrightarrow{T} \quad \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\Sigma = T^2.$$



$$\Sigma = T \cdot T^{t}$$