① Define norms

② Examples "common" norms

③ Properties of norms (Cauchy-Schwartz)

Axler 6A

④ Hamming distance, error correcting codes

Wiki,

Def If V is an inner product space with inner product $\langle \cdot , \cdot \rangle$ the norm of $v \in V$ is

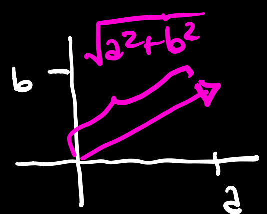
$$\|v\| = \sqrt{\langle v, v \rangle}$$

Ex 1. $\mathbb{R}$, $\langle a, b \rangle = ab$

$$\|a\| = \sqrt{\langle a, a \rangle} = \sqrt{a^2} = |a|$$

Ex 2. $\mathbb{R}^2$, $\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 a_2 + b_1 b_2$

$$\|(a,b)\| = \sqrt{a \cdot a + b \cdot b} = \sqrt{a^2 + b^2} = \text{"radius of } (a,b)\text{"}$$



Ex 3. $\mathbb{C}$, $\langle a+bi, c+di \rangle = (a+bi)(c-di)$

$$\|a+bi\| = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2} = \underbrace{\text{"radius"}}_{\text{modulus.}}$$

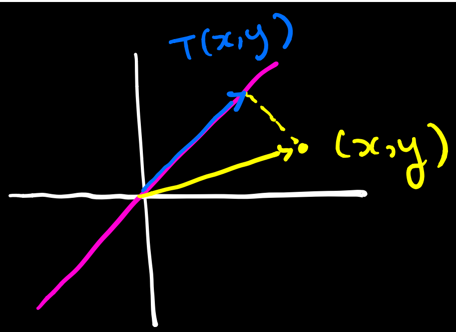
Ex 4. $\mathbb{R}^n$, $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$.

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Ex 5. $C^0([-1,1])$, $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$.

$$\|f\| = \left(\int_{-1}^{1} f(x)^2\, dx \right)^{1/2}$$

ASIDE.



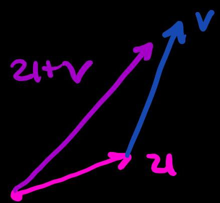
$T(x,y)$ is the closest point on the pink line to (x,y).

NOTE ① $\|v\| = 0 \iff v = \vec{0}$. (why?)

② $\|\lambda v\| = |\lambda| \, \|v\|$. (Why?)

Thinking of norms as lengths of paths, there is a fundamental property we would like:

$$(*) \quad \|u + v\| \leq \|u\| + \|v\|.$$



How do we show $(*)$?

Use the $\boxed{\text{Cauchy-Schwartz inequality}}$

For simplicity, take $F = \mathbb{R}$.

$\forall\, u, v \in V$

$$\langle u, v\rangle \cdot \langle u, v\rangle \leq \langle u, u\rangle \cdot \langle v, v\rangle$$

Proof. Let $t \in \mathbb{R}$

$$0 \leq \|u + tv\|^2 = \langle u + tv,\ u + tv\rangle$$

$$= \langle u, u\rangle + 2\langle u, tv\rangle + \langle tv, tv\rangle$$

$$= \langle u, u\rangle + 2\langle u, v\rangle t + \langle v, v\rangle t^2 = p(t)$$

This is a polynomial in t. Its roots occur at

$$\text{“}\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\text{”} = \frac{-2\langle u, v\rangle \pm \sqrt{4\langle u, v\rangle^2 - 4\langle v, v\rangle\langle u, u\rangle}}{2\langle v, v\rangle}$$

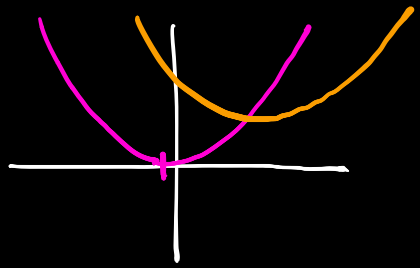
$$\underbrace{at^2 + bt + c}$$

$p(t)$ has two roots precisely when

$$4\langle u,v\rangle^2 - 4\langle v,v\rangle\langle u,u\rangle > 0 .$$

$$\langle u,v\rangle^2 > \langle v,v\rangle\langle u,u\rangle .$$

Because $0 \le p(t)$, $p(t)$ has at most one root.



I conclude that $\langle u,v\rangle^2 \le \langle v,v\rangle\langle u,u\rangle$.

Prop. $\forall\ u, v \in V \quad \|u+v\| \leq \|u\| + \|v\|.$

Pf. The statement of the proposition is equivalent to $\|u+v\|^2 \leq (\|u\| + \|v\|)^2 \iff$

$$\langle u+v, u+v\rangle \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \iff$$

$$\cancel{\langle u,u\rangle} + 2\langle u,v\rangle + \cancel{\langle v,v\rangle} \leq \cancel{\langle u,u\rangle} + 2\|u\|\|v\| + \cancel{\langle v,v\rangle} \iff$$

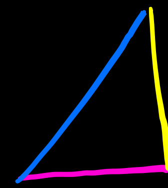
$$\langle u,v\rangle \leq \|u\|\cdot\|v\| \iff$$

$$\langle u,v\rangle^2 \leq \langle u,u\rangle\cdot\langle v,v\rangle$$

This is true by the Cauchy-Schwartz inequality. $\square$

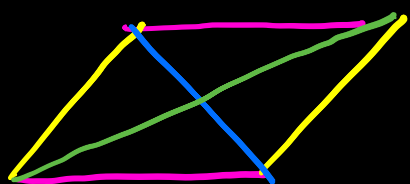
Pythagoras' Law. We call two vectors $u, v \in V$ **orthogonal** if $\langle u, v \rangle = 0$. If so,

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$



Parallelogram Law. For any $u, v \in V$

$$\|u+v\|^2 + \|u-v\|^2 = 2\left[\|u\|^2 + \|v\|^2\right].$$



Upshot Abstract norms behave like the Euclidean norm.

Here's an interesting norm:

$F = \mathbb{F}_2 = \{0, 1\}$

$V = F^n$

Elements of V look like "bit strings of length n":

i.e. $\underbrace{0 0 1 1 0 1 1 0 \cdots 0 1 1 1 0}_{n \text{ elements}}$

Hamming weight: $\|s\| =$ # non-zero entries in s
(the # of '1's)

e.g. $\| 0110 \| = 2$.

This defines a notion of distance: the distance between s and t is $\|s-t\|$.

HAMMING DISTANCE.