OUR GOAL. Find a basis of eigenvectors.

Prop. If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of $T: V \to V$, with eigenvectors $v_1, \ldots, v_n$, then $\{v_1, \ldots, v_n\}$ is linearly independent.

Why might this be true??

Simple case: λ_1, λ_2 distinct. v_1, v_2 eigenvectors.

If $v_1 = c \cdot v_2$ then

$$\lambda_1 v_1 = T(v_1) = T(c \cdot v_2) = c \cdot T(v_2) = c \cdot \lambda_2 v_2.$$

$\lambda_1 = 0 \rightsquigarrow \lambda_2 = 0 \rightsquigarrow$ contradicts distinction.

$\lambda_1 \neq 0 \rightsquigarrow v_1 = \frac{c \cdot \lambda_2}{\lambda_1} v_2$.

$c \cdot \frac{\lambda_2}{\lambda_1} = c \rightsquigarrow \lambda_2 = \lambda_1$

$\rightsquigarrow$ contradiction.

Next simplest case: $\lambda_1, \lambda_2, \lambda_3$ distinct. v_1, v_2, v_3 eigenvectors.

no two are scalar multiples.

Let's suppose $v_1 = a \cdot v_2 + b \cdot v_3$, a, b non-zero.

$$\lambda_1 v_1 = T(v_1) = T(a \cdot v_2 + b \cdot v_3) = \lambda_2 a v_2 + \lambda_3 b v_3$$

$\boxed{\text{TRICK}}$ Two equations in 3 unknowns... can I simplify?

$$\begin{aligned} [\lambda_1 v_1 &= \lambda_1 a \cdot v_2 + \lambda_1 b \cdot v_3] \\ - \quad \lambda_1 v_1 &= \lambda_2 a \cdot v_2 + \lambda_3 b \cdot v_3 \\ \hline 0 &= a \cdot (\lambda_1 - \lambda_2) v_2 + b(\lambda_1 - \lambda_3) v_3 . \end{aligned}$$

$\rightsquigarrow$ $\lambda_1 - \lambda_2 = 0$
$\lambda_1 - \lambda_3 = 0$.

$\rightsquigarrow$ $\lambda_1 = \lambda_2 = \lambda_3$

$\rightsquigarrow$ contradiction.

TRICK SCALES EASILY!

Pf of Prop By induction.

Base case ✓

Inductive step: Suppose we've proved that $\{v_1, \ldots, v_k\}$ is linearly independent.

Assume for contradiction $v_{k+1} \in \operatorname{span}\{v_1, \ldots, v_k\}$.

Write $\boxed{v_{k+1} = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k.}$

Apply T: $\lambda_{k+1} v_{k+1} = T(v_{k+1}) = a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \ldots + a_k \lambda_k v_k$

$$\begin{array}{rl} & \lambda_{k+1} v_{k+1} = \lambda_{k+1} a_1 v_1 + \lambda_{k+1} a_2 v_2 + \ldots + \lambda_{k+1} a_k v_k \\ - & \lambda_{k+1} v_{k+1} = a_1 \lambda_1 v_1 + \ldots + a_k \lambda_k v_k \\ \hline & 0 = a_1(\lambda_{k+1} - \lambda_1) v_1 + \ldots + a_k(\lambda_{k+1} - \lambda_k) v_k \end{array}$$

$\Rightarrow a_1 = 0, a_2 = 0, \ldots, a_k = 0.$

$\Rightarrow v_{k+1} = 0$, contradicting that it is an eigenvector. $\square$

__COR__ If $T \in \mathcal{L}(V)$ has $\dim(V)$ distinct eigenvalues, then V has a basis made up of eigenvectors of T.

IDEA: to understand eigenvectors, study eigenvalues.

DIGRESSION: fields

Characterized by 2 operations: $+$, and $\cdot$

Quintessential example: $\mathbb{R}$, with the usual $+$ and $\cdot$.

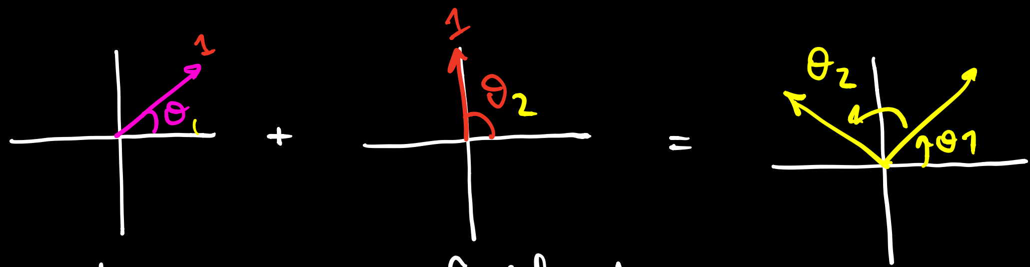
Q $\mathbb{R}^2$, with the usual $+$, what multiplicative structures does it have?

① $(x,y)\cdot(w,z) = (x\cdot w,\ y\cdot z)$

Does this give a field structure? NO

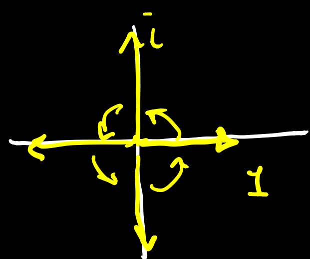
$(x,0)\cdot(x^{-1},0) = \underbrace{(1,0)}_{(1,1)}$ $\qquad (x,0)\cdot(x^{-1},1) = (1,0)$

② Writing in polar coordinates, $(r_1, \theta_1)\cdot(r_2, \theta_2) = (r_1 r_2, \theta_1+\theta_2)$



This does give a field structure.

This gives the same structure as complex multiplication.



$i \cdot 1 = i$

$i \cdot i = -1$

$i \cdot (-1) = -i$

$i \cdot (-i) = 1$

③ Anything else . . .

$\boxed{Q}$ For which n does $\mathbb{R}^n$, with the usual addition, admit a field structure?

Quaternions : Hamilton

What else might I want?

I want my fields to be <u>algebraically closed</u>: every polynomial has a root.

This is false for $\mathbb{R}$! $p(x) = x^2 + 1$

This is false for <u>finite fields</u>.

[digression: prove that there are ∞ many prime #s!

Assume for contradiction that there aren't.

$\{p_1, p_2, \ldots, p_n\}$ are all of the prime #s.

consider the # $N = \underbrace{p_1 \cdot p_2 \cdot \ldots \cdot p_n}_{} + \underbrace{1}_{}$.

Claim: N is prime.

If $a_1, \dots, a_n$ are the elements of my field consider $p(x) = (x-a_1)(x-a_2)\cdots(x-a_n) + 1$.

$p(a_i) \neq 0$.

So $p(x)$ has no roots.

$\underbrace{\text{However, } \mathbb{C} \text{ is algebraically closed}}$.

$\underbrace{\text{Fundamental}}_{\text{hard}}$ Theorem of $\underbrace{\text{Algebra}}_{\text{requires analysis!}}$

<u>Note</u>. If $p(\lambda) = 0$ then $p(z) = (z-\lambda)\cdot q(z)$ for some polynomial $q(z)$.

[ASIDE: proof uses the fact that polynomial rings are something called Euclidean domains]

Thm. Every polynomial $p(z) \in \mathcal{P}(\mathbb{C})$ can be factored as $p(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$.

Pf Sketch. We know that $\exists\ \lambda_1$ with $p(\lambda_1) = 0$.

Note $\rightsquigarrow$ we can write $p(z) = (z-\lambda_1)\cdot q(z)$.

$\deg(q) < \deg(p)$.

Use induction on the degree of a polynomial.

BACK TO EIGENVALUES

Does $T \in \mathcal{L}(V)$ even have an eigenvalue?

Ex. $\mathbb{R}^2 \to \mathbb{R}^2$ did not have any eigenvalues.
$(x,y) \mapsto (-y,x)$

$T - \underbrace{\lambda I_V}_{}$.

$v \mapsto \lambda v$.

Prop. If V is finite-dimensional, $T \in \mathcal{L}(V)$, $\lambda \in F$ then λ is an eigenvalue of $T \iff$

$T - \lambda I_V$ is NOT injective $\iff$
$T - \lambda I_V$ is NOT surjective $\iff$
$T - \lambda I_V$ is NOT bijective.

Rank-nullity thm. T is a map from V to V.

Pf. $v \in V$ non-zero

$$T(v) = \lambda \cdot v \iff (T - \lambda \cdot I_v)(v) = T(v) - \lambda I_v(v)$$
$$= T(v) - \lambda v$$
$$= 0.$$

$\square$

Recall that we defined $T^m = \underbrace{T \circ T \circ \ldots \circ T}_{m \text{ times}}$

T^m can be useful to study. For example, if λ is an eigenvalue of T, then λ^m is an eigenvalue of T^m.

We can define polynomials in T:

$$p(T) = a_0 + a_1 T + a_2 T^2 + \ldots + a_k T^k.$$

$p(T) \in \mathcal{L}(V)$

We will use this fact to prove:

<u>Thm</u> If $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional, then any operator $T \in \mathcal{L}(V)$ has an eigenvalue.