1. Define orthonormal basis

2. Prove existence (Gram-Schmidt process)

$\left.\right\}$ 6B

3. Minimizing distance $\leftarrow$ 6C

Def. Two vectors v, w are *orthogonal* if $\langle v, w \rangle = 0$.

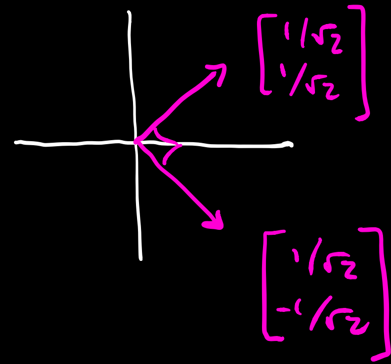
Def. A list of vectors is *orthonormal* if every vector in the list has norm 1 and is orthogonal to every other vector in the list.

In other words, $\{e_1, \ldots, e_k\}$ is orthonormal if

$$\|e_i\| = 1 \quad \text{and} \quad \langle e_i, e_j \rangle = 0 \quad \text{whenever } i \neq j.$$

Ex. $\left\{ \underset{\overset{\shortparallel}{e_1}}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \underset{\overset{\shortparallel}{e_2}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \ldots, \underset{\overset{\shortparallel}{e_k}}{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} \right\}$ is an orthonormal list in $\mathbb{R}^n$.

Ex. $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal list in $\mathbb{R}^2$.



Ex. $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal list in $\mathbb{R}^2$.

Prop. An orthonormal list is linearly independent.

To prove, we need a

Lemma If $\{e_1, \ldots, e_n\}$ is an orthonormal list then $\|a_1 e_1 + \ldots + a_n e_n\|^2 = |a_1|^2 + \ldots + |a_n|^2$ for scalars $a_1, \ldots, a_n$.

Pf. Base case: $\|a_1 e_1\|^2 = |a_1|^2 \|e_1\|^2$ (homogeneity)

$= |a_1|^2 \cdot 1^2$ (by assumption that

$= |a_1|^2$ $\{e_1\}$ is

orthonormal)

Inductive step: Assume that the Lemma holds for

$\{e_1, \ldots, e_{n-1}\}$.

Note: $\langle \underbrace{a_n e_n}_{u}, \underbrace{a_1 e_1 + \ldots + a_{n-1} e_{n-1}}_{v} \rangle = 0$ (homogeneity, additivity)

Pythagoras' Law:

$$\|(a_n e_n) + (a_1 e_1 + \ldots + a_{n-1} e_{n-1})\|^2 =$$

$$\underbrace{\|a_n e_n\|^2} + \underbrace{\|a_1 e_1 + \ldots + a_{n-1} e_{n-1}\|^2}.$$

$$= |a_n|^2 \|e_n\|^2 + |a_1|^2 + \ldots + |a_{n-1}|^2$$

$= |a_n|^2 + |a_1|^2 + \dots + |a_{n-1}|^2$

$\square$

Homogeneity: $\langle c v, w \rangle = c \langle v, w \rangle$

c: scalar

<u>Proof of Prop.</u> Suppose $a_1 e_1 + \dots + a_n e_n = 0$.

By the Lemma, $\|a_1 e_1 + \dots + a_n e_n\|^2 = \|0\|^2$

$$|a_1|^2 + \dots + |a_n|^2 = 0$$

$\Rightarrow |a_i|^2 = 0 \quad \forall i$

$\Rightarrow a_i = 0 \quad \forall i.$

$\square$

AMAZING! NORMS DETECT LINEAR INDEPENDENCE!

Other proof: If $0 = a_1 e_1 + \dots + a_n e_n$ then

$$\langle 0, e_i \rangle = \langle a_1 e_1 + \dots + a_n e_n, e_i \rangle = \langle a_i e_i, e_i \rangle$$

$$\langle 0, e_i \rangle = 0 \qquad \langle a_i e_i, e_i \rangle = a_i.$$

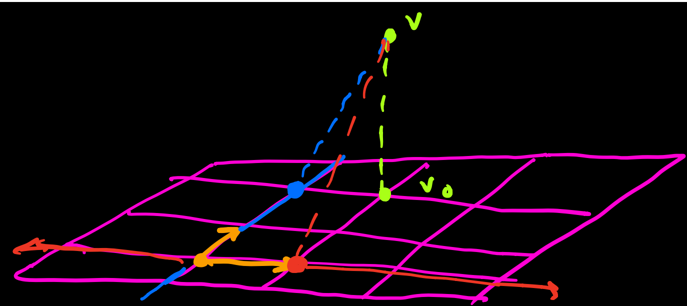
$\rightsquigarrow \; 0 = a_i.$

$\square$

Cor. Orthonormal lists of length $\dim(V)$ are bases.

Prop. If $\{e_1, \dots, e_n\}$ is an orthonormal list and $v \in V$ then $v_0 = \sum_{i=1}^{n} \langle v, e_i \rangle e_i$ is the closest vector in $\mathrm{span}\{e_1, \dots, e_n\}$ to v.

Fourier formula.



Pf. $\langle v_0, e_i \rangle = \left\langle \sum_{j=1}^{n} \langle v, e_j \rangle e_j , e_i \right\rangle = \langle v, e_i \rangle$

$\langle v - v_0, e_i \rangle = 0$

$\Rightarrow$ $v - v_0$ is orthogonal to all $w \in \operatorname{span}\{e_1, \dots, e_n\}$

useful trick

$$\|v - w\|^2 = \|\underbrace{(v - v_0)}_{\text{orthogonal to span}\{e_1,\dots,e_n\}} + \underbrace{(v_0 - w)}_{\in\, \text{span}\{e_1,\dots,e_n\}}\|^2$$

$$= \|v - v_0\|^2 + \underbrace{\|v_0 - w\|^2}_{\checkmark}$$ By Pythagoras' Law

$$\geq \|v - v_0\|^2.$$

So w is "farther away from v" than v_0. $\square$

Do orthonormal bases exist? Yes.

GRAM-SCHMIDT PROCESS: turns a basis $\{f_1, \dots, f_n\}$ into an orthonormal basis $\{e_1, \dots, e_n\}$.

We will do this so that $\operatorname{span}(f_1) = \operatorname{span}(e_1)$

$$\operatorname{span}(f_1, f_2) = \operatorname{span}(e_1, e_2)$$

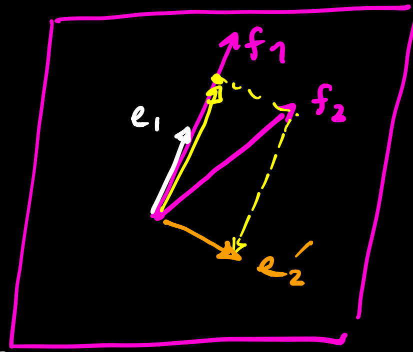
$$\vdots$$

1. $e_1 = \dfrac{f_1}{\|f_1\|}$. $\quad \|e_1\|^2 = \left\langle \dfrac{f_1}{\|f_1\|}, \dfrac{f_1}{\|f_1\|} \right\rangle = \dfrac{1}{\|f_1\|^2} \langle f_1, f_1 \rangle$

$$= \|f_1\|^2 / \|f_1\|^2 = 1.$$

2. $e_2' = f_2 - \langle f_2, e_1 \rangle e_1$

3. $e_2 = \dfrac{e_2'}{\|e_2'\|}$.



4. $e_3' = f_3 - \langle f_3, e_1 \rangle e_1 - \langle f_3, e_2 \rangle e_2$

5. $e_3 = \dfrac{e_3'}{\|e_3'\|}$.

$\vdots$

<u>Pf</u>. Exercise or read in Axler.

<u>Wed</u>. Approximating functions via polynomials and linear algebra.