1 Lecture 1: Homological algebra

$\mathbb{K}$ a field.

1.1 A_∞ categories

DEFINITION 1 (A_∞ CATEGORY) An A_∞ category $\mathcal{A}$ consists of:

1. Objects $Ob(\mathcal{A})$
2. Graded $\mathbb{K}$-vector-spaces $hom_{\mathcal{A}}(X, Y)$ for $X, Y \in Ob(\mathcal{A})$.
3. A_∞-morphisms
   - $\mu^1_{\mathcal{A}} : hom_{\mathcal{A}}(X, Y) \to hom_{\mathcal{A}}(X, Y)[1]$,
   - $\mu^2_{\mathcal{A}} : hom_{\mathcal{A}}(Y, Z) \otimes hom_{\mathcal{A}}(X, Y) \to hom_{\mathcal{A}}(X, Z)[0]$,

     $\vdots$
   - $\mu^k_{\mathcal{A}} : hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes ... \otimes hom_{\mathcal{A}}(X_0, X_1) \to hom_{\mathcal{A}}(X_0, X_k)[2-k]$.
4. A_∞ relations
   - $\mu^1(\mu^1(\cdot)) = 0$,
   - $\mu^1(\mu^2(\cdot, \cdot)) \pm \mu^2(\mu^1(\cdot), \cdot) \pm \mu^2(\cdot, \mu^1(\cdot)) = 0$,

     $\vdots$

EXAMPLE 1) A_∞ algebras are precisely the A_∞ categories with one element.

EXAMPLE 2) dg-categories become A_∞ categories by setting $\mu^k = 0$ for $k \geq 3$.

We can define $H^*(\mathcal{A})$, which has the same objects as $\mathcal{A}$ and morphisms $H^*(hom_{\mathcal{A}}(X, Y))$, and $H^0(\mathcal{A})$, which has the same objects as $\mathcal{A}$ and the 0-degree morphisms of $H^*(\mathcal{A})$. μ^2 gives an associative product on $H^*(\mathcal{A})$.

DEFINITION 2 (A_∞ FUNCTOR) An A_∞ functor consists of the action on objects $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and maps

$$\mathcal{F}^d : hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes ... \otimes hom_{\mathcal{A}}(X_0, X_1) \to hom_{\mathcal{A}}(\mathcal{F}(X_0), \mathcal{F}(X_d))[1-d]$$

satisfying "an appropriate homomorphism equation." Eg

$$\mu^1(\mathcal{F}^1(\cdot)) \pm \mathcal{F}^1(\mu^1(\cdot)) = 0$$

and

$$\mu^2(\mathcal{F}^1(\cdot), \mathcal{F}^1(\cdot)) \pm \mu^1(\mathcal{F}^2(\cdot, \cdot)) \pm \mathcal{F}^1(\mu^2(\cdot, \cdot)) \pm \mathcal{F}^2(\mu^1(\cdot), \cdot) \pm \mathcal{F}^2(\cdot, \mu^1(\cdot)) = 0.$$

Suppose $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ induces an isomorphism between $H^*(\mathcal{A})$ and $H^*(\mathcal{B})$. Then $\exists \mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F} \circ \mathcal{G}$ induces $Id_{H(\mathcal{B})}$ and $\mathcal{G} \circ \mathcal{F}$ induces $Id_{H(\mathcal{A})}$. Thus, we have a good definition for quasi-isomorphic categories (unlike the dg case).

DEFINITION 3 Two objects are <u>quasi-isomorphic</u> if they are isomorphic in $H^0(\mathcal{A})$.

DEFINITION 4 Let $X, Y \in Ob(\mathcal{A})$. X is a <u>deformation retract</u> of Y if $\exists\ i \in hom_{\mathcal{A}}(X, Y)$ and $\pi \in hom_{\mathcal{A}}(Y, X)$ such that $\pi \circ i \cong Id_X$.

1. Any A_∞-category is quasi-isomorphic to a dg-category.
2. $\mathcal{A}$ is minimal if $\mu^1_{\mathcal{A}} = 0$. Any A_∞ category is quasi-isomorphic to a minimal one.
3. $\mathcal{A}$ is proper if $H^*(\hom_{\mathcal{A}}(X,Y))$ is finite total dimensional for all X, Y.
4. There are also notions of cyclicity and unitality, which we may talk about later.

We assume that $\mathcal{A}$ is cohomologically unital, that is, that $H^*(\hom(X,Y))$ has a unit which is well-behaved with respect to $[\mu^2]$.

1.2 Twisted complexes

$\mathcal{A} \subset \mathcal{A}^{tw}$. Construct $\mathcal{A}^{tw}$ from $\mathcal{A}$ by defining shifts $X \to X[k], k \in \mathbb{Z}$ and cones: for $X_0, X_1 \in \mathrm{Ob}(\mathcal{A})$ and $a \in \hom_{\mathcal{A}}(X_0, X_1)$ such that $\mu_1(a)$,

$$\mathrm{Cone}(a) = \left\{ X_0[1] \oplus X_1, \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \right\}.$$

$D^{tw} := H^0(\mathcal{A}^{tw})$ is triangulated.

1.3 A_∞ modules

$\mathcal{A}^{mod} := \mathrm{Fun}(\mathcal{A}^{opp}, \mathrm{Ch})$, where Ch is the dg-category of chain complexes over $\mathbb{K}$. Concretely, a right-$\mathcal{A}$-module M is defined by

1. A graded $\mathbb{K}$-vector-space $M(X)$ for each $X \in \mathrm{Ob}(\mathcal{A})$,
2. operations

$$\begin{aligned}
&\mu_M^{1:0} : M(X) \to M(X)[1], \\
&\mu_M^{1:1} : M(Y) \otimes \hom(X,Y) \to M(X), \\
&\mu_M^{1:2} : M(Z) \otimes \hom(Y,Z) \otimes \hom(X,Y) \to M(X)[-1], \\
&\vdots \\
&\mu_M^{1:k} M(X_k) \otimes \hom(X_{k-1}, X_k) \otimes ... \otimes \hom(X_0, X_1) \to M(X_0)[1-k],
\end{aligned}$$

3. and relations

$$\begin{aligned}
&\mu^{1:0}(\mu^{1:0}(\cdot)) = 0, \\
&\mu^{1:0}(\mu^{1:1}(m,a)) \pm \mu^{1:1}(\mu^{1:0}(m), a) \pm \mu^{1:1}(m, \mu^1(a)) = 0, \\
&\vdots
\end{aligned}$$

.

Note that the relations allow us to define $H^*(M(X))$.

Definition 5 (Morphism of A_∞ modules) Morphisms of A_∞ = natural transformations.

$$f_n : M(X_n) \otimes \hom(X_{n-1}, X_n) \otimes ... \otimes \hom(X_0, X_1) \to N(X_0)[1-n]$$

such that

$$\sum_{\#\mathrm{inputs}} \pm f_k(\mathrm{Id}^{\otimes} \otimes \mu^\ell \otimes \mathrm{Id}^{\otimes}) \pm f_k(\mu^{1:\ell} \otimes \mathrm{Id}^{\otimes}) = \sum_{\#\mathrm{inputs}} \mu^{1:i}(f_j \otimes \mathrm{Id}^{\otimes}).$$

Fact: If $H^*(M(X)) = 0 \;\forall\, X \in Ob(\mathcal{A})$, then $M \overset{q.i}{\cong} 0$.

DEFINITION 6 $D^{mod} = H^0(\mathcal{A}^{mod})$. This is a triangulated and Karoubi complete category. ie. given $M \in \mathcal{A}^{mod}$ and $[\pi] \in H^0(hom_{\mathcal{A}^{mod}}(M, M))$ idempotent, $\exists\, \overline{M^{[\pi]} \in Ob(\mathcal{A}^{mod})}$ which is the homotopy retract of M associated to π.

DEFINITION 7 (YONEDA EMBEDING) The Yoneda embedding is a map $\mathcal{A} \to \mathcal{A}^{mod}$ sending $Y \in Ob(\mathcal{A})$ to $hom_{\mathcal{A}}(-, Y)$.

EXAMPLE 3) If $\mathcal{A}$ is an A_∞ algebra then $\mathcal{A}^{Yon}$ is the free module $\mathcal{A}$.

DEFINITION 8 M is perfect if it is a homotopy retract of the Yoneda image of $\mathcal{A}^{tw}$. The category of perfect $\mathcal{A}$ modules is $\mathcal{A}^{perf}$. We define $\mathcal{D}^{perf} = H^0(\mathcal{A}^{perf})$. This latter category is triangulated and Karoubi complete.

DEFINITION 9 M is proper if $H^*(M(X))$ has finite total dimension for each X.

LEMMA 1 *If $\mathcal{A}$ is a proper A_∞ category then all $M \in \mathcal{A}^{perf}$ are proper.*

LEMMA 2 *If $M, N \in \mathcal{A}^{mod}$ such that M is perfect and N proper, then $H^k(hom_{\mathcal{A}^{mod}}(M, N))$ is finite dimensional.*

Suppose $\mathcal{A}$ is an A_∞ algebra and $M \in \mathcal{A}^{perf}$ is built from finitely many copies of $\mathcal{A}$. Let $||M||$ be the minimal number of such copies. Then

$$\dim H^k(hom_{\mathcal{A}^{mod}}(M, N)) \leq \dim H^k(N) \cdot ||M||.$$

1.4 A_∞ bimodules

$(\mathcal{A}, \mathcal{B})^{mod} := (\mathcal{A}^{opp} \otimes \mathcal{B})^{mod}$.

DEFINITION 10 An $(\mathcal{A}, \mathcal{B})^{mod}$ bimodule Q is

1. a collection of graded vector spaces $\{Q((X, Y))\}$ over all $(X, Y) \in Ob(\mathcal{B}) \times Ob(\mathcal{A})$,
2. operations $\mu_Q^{n:1:s} : hom_{\mathcal{A}}(Y_{n-1}, Y_n) \otimes ... \otimes hom_{\mathcal{A}}(Y_0, Y_1) \otimes Q(X_s, Y_0) \otimes hom_{\mathcal{B}}(X_{s-1}, X_s) \otimes ... \otimes hom_{\mathcal{B}}(X_0, X_1) \to Q(X_0, Y_n)[1 - n - s]$, and
3. relations

$$\sum \left[\pm\mu_Q^{*:1:*}(..., \mu_Q^{*:1:*}(..., \cdot, ...), ...)\right] + \sum \left[\pm\mu_Q^{*:1:*}(..., \mu_{\mathcal{A}}^{*}(...), ..., \cdot, ...)\right] + \sum \left[\pm\mu_Q^{*:1:*}(..., \cdot, ..., \mu_{\mathcal{B}}^{*}(...), ...)\right] = 0.$$

EXAMPLE 4) Diagonal $(\mathcal{A}, \mathcal{A})$ module, $Q(X, Y) = hom_{\mathcal{A}}(X, Y)$, and μ_Q is inherited.

EXAMPLE 5) $\mathcal{F} \in Fun(\mathcal{A}, \mathcal{B})$ yields the graph bimodule $Q = Graph(\mathcal{F})$, where $Q(X, Y) = hom_{\mathcal{B}}(X, \mathcal{F}(Y))$.

$(\mathcal{A}, \mathcal{B})^{mod}$ is a d.g. category equal to $Fun((\mathcal{A}^{opp} \otimes \mathcal{B})^{opp}, Ch)$.

2 Lecture 2: Homological Algebra & Hochschild homology

DEFINITION 11 (CONVOLUTION) Let $\mathcal{P} \in (\mathcal{A}, \mathcal{B})^{mod}$. $\mathcal{P}$ gives rise to the convolution functor

$$\begin{aligned} \Phi_{\mathcal{P}} : \mathcal{A}^{mod} &\to \mathcal{B}^{mod} \\ M &\mapsto \text{“}M \otimes_{\mathcal{A}} P\text{''} \end{aligned}$$

where

$$M \otimes \mathcal{P}(X) = \left[\bigoplus_{Y_0 \in Ob(\mathcal{A})} M(Y_0) \otimes_{\mathbb{K}} \mathcal{P}(X, Y_0) \right] \oplus \left[\bigoplus_{Y_0, Y \in Ob(\mathcal{A})} M(Y_1) \otimes_{\mathbb{K}} hom(Y_0, Y_1) \otimes_{\mathbb{K}} \mathcal{P}(X, Y_0)[1] \right] \ldots$$

Remarks

1. $$\Phi_{\mathcal{A}} \cong Id_{\mathcal{A}^{mod}}$$

   ($\mathcal{A}$ the diagonal bimodule.)

2. There is a diagram, commutative up to quasi-isomorphism of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{F}} & \mathcal{B} \\ \downarrow{\scriptstyle Yon} & & \downarrow{\scriptstyle Yon} \\ \mathcal{A}^{mod} & \xrightarrow[\Phi_{Graph(\mathcal{F})}]{} & \mathcal{B}^{mod} \end{array}$$

3. If $\mathcal{P} = (X^{opp})^{Yon} \otimes Y^{Yon}$ then $\Phi_{\mathcal{P}}(M) = M(X) \otimes Y^{Yon}$.

4. Consequence: If $\mathcal{A}$ is proper and $\mathcal{P}$ is perfect, then $\Phi_{\mathcal{P}}$ maps proper modules to perfect ones.

DEFINITION 12 $\mathcal{A}$ is smooth if $(\mathcal{A}, \mathcal{A})^{mod}$ is perfect.

LEMMA 3 $\mathcal{A} = K[X]$, X *an algebraic variety* $\implies$ *the two notions of smoothness coincide.*

LEMMA 4 $\mathcal{A}$ *smooth and proper* $\implies$ $\mathcal{A}^{prop} = \mathcal{A}^{perf}$.

PROOF: Apply $\phi_{\mathcal{A}}$ to proper M and use Remarks 1 and 4. ▶

2.1 Quotient categories

$\mathcal{B} \subset \mathcal{A}$ a full subcategory. Want to define $\mathcal{C} = \mathcal{A}/\mathcal{B}$. $\mathcal{C}$ is equipped with a map $Q : \mathcal{A} \to \mathcal{C}$ such that the composition $\mathcal{B} \hookrightarrow \mathcal{A} \to \mathcal{C}$ is essentially 0; and such that, if $\mathcal{D}$ is an A_∞ category, then $Fun(\mathcal{C}, \mathcal{D}) \to Fun(\mathcal{A}, \mathcal{D})$ is cohomologically full and faithful, with image precisely those functors which kill $\mathcal{B}$.

THEOREM 1 *These exist.*

3 Hochschild Homology

3.1 Motivation

X closed, symplectic.

$$\begin{array}{ccc} QH^* & & \\ \uparrow & \nwarrow & \\ HH_*(\mathrm{Fuk}(X)) & \longleftarrow & HH_*(\mathcal{A}) \end{array}$$

Suppose $\mathcal{A} \hookrightarrow \mathrm{Fuk}(X)$. If Id lies in the image then "we can study $\mathcal{A}$ instead of $\mathrm{Fuk}(X)$."

3.2 Construction

3.2.1

Suppose $\mathcal{A}$ is an algebra over a field $\mathbb{K}$. For an $(\mathcal{A}, \mathcal{A})$-bimodule M (which is a right $\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}^{opp}$-module), we define

$$HH_n(\mathcal{A}, M) := \mathrm{Tor}_n^{\mathcal{A}^e}(M, \mathcal{A}).$$

To compute, take the standard bar resolution:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathcal{A}^{\otimes 4} & \longrightarrow & \mathcal{A}^{\otimes 3} & \longrightarrow & \mathcal{A}^{\otimes 2} & \longrightarrow & 0 \\ & & \alpha \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \end{array}$$

where, for example, $\alpha : a \otimes b \otimes c \otimes d \mapsto ab \otimes c \otimes d - a \otimes bc \otimes d + a \otimes b \otimes cd$. Tensor with M to get

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & M \otimes_{\mathcal{A}^e} \mathcal{A}^{\otimes 4} & \longrightarrow & M \otimes_{\mathcal{A}^e} \mathcal{A}^{\otimes 3} & \longrightarrow & M \otimes_{\mathcal{A}^e} \mathcal{A}^{\otimes 2} & \longrightarrow & 0 \\ & & \alpha(\cong) \downarrow & & \cong \downarrow & & \cong \downarrow & & \\ \ldots & \longrightarrow & M \otimes_{\mathbb{K}} \mathcal{A}^{\otimes 2} & \longrightarrow & M \otimes_{\mathbb{K}} \mathcal{A} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where $\alpha : m \otimes a \otimes b \otimes c \otimes d \mapsto dmab \otimes c - dma \otimes bc + cdma \otimes b$. Thus, we may define $HC_n(\mathcal{A}, M) := M \otimes_{\mathbb{K}} \mathcal{A}^{\otimes n}$ with differential

$$\partial(m \otimes a_1 \otimes \ldots \otimes a_n) = ma_1 \otimes a_2 \otimes \ldots \otimes a_n + \left[\sum_{1 \leq i < n} \pm m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n \right] \pm a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}$$

and set

$$HH_n(\mathcal{A}, M) = H_n(HC_*(\mathcal{A}, M), \partial).$$

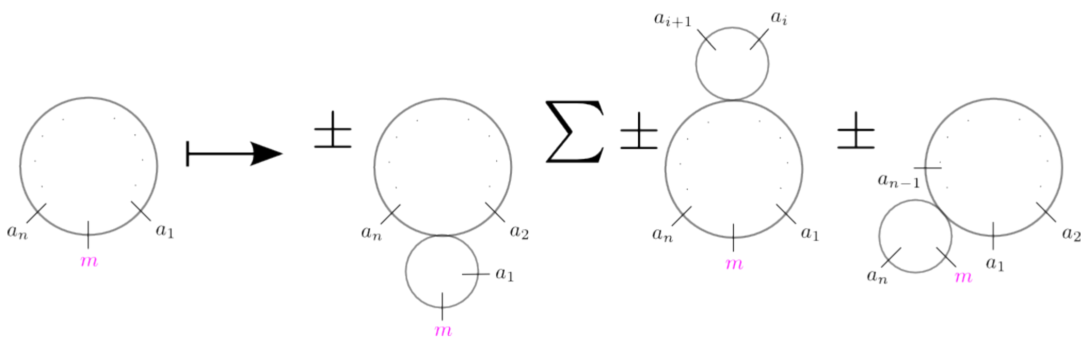


Figure 1: An illustration of the differential

3.2.2

Suppose $\mathcal{A}$ is now an A_∞ algebra over $\mathbb{K}$. Let M be an $(\mathcal{A}, \mathcal{A})$-bimodule. We again have

$$\mathrm{HH}_*(\mathcal{A}, M) := \mathrm{Tor}_*^{\mathcal{A}^e}(M, \mathcal{A}).$$

The differential is modified according to Figure 2.

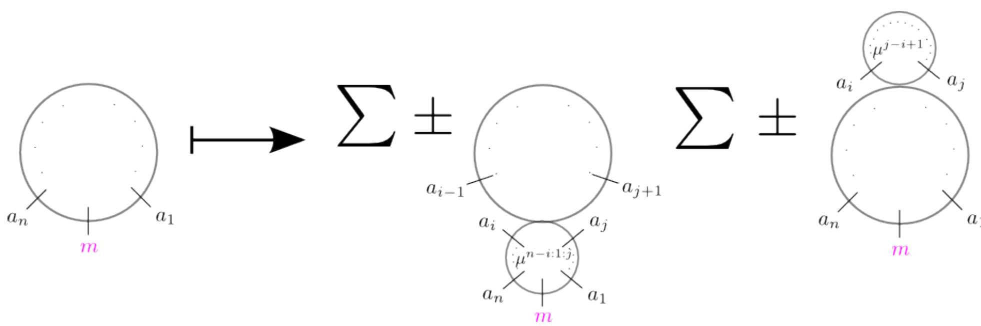


Figure 2: A_∞ algebra differential

3.2.3

Now let $\mathcal{A}$ be an A_∞-category and Q an $(\mathcal{A}, \mathcal{A})$ bimodule. The Hochschild chain complex is

$$\mathrm{HC}_{(}\mathcal{A}, Q) := \bigoplus Q(X_d, X_0) \otimes_{\mathbb{K}} \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \ldots \otimes \hom_{\mathcal{A}}(X_0, X_1)[d].$$

Less concretely,

$$\mathrm{HH}_*(\mathcal{A}, Q) = H^*(Q \otimes_{\mathcal{A}^e} \mathcal{A}) \tag{1}$$

3.3 Key Properties

1. Covariant functoriality in Q (obvious from Eq.1).
2. Given an A_∞-functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and a $(\mathcal{B}, \mathcal{B})$-bimodule Q, we get a map $\mathrm{HH}_*(\mathcal{A}, \mathcal{F}^*(Q)) \to \mathrm{HH}_*(\mathcal{B}, Q)$, where $\mathcal{F}^*$ is the pullback on both sides.

(a) Notation: $HH_*(\mathcal{A}) := HH_*(\mathcal{A}, \mathcal{A})$ (the latter $\mathcal{A}$ refers to the diagonal bimodule).

(b) 1 and 2 imply $HH_*(\mathcal{A})$ is covariantly functorial.

3. Morita invariance: The Yoneda embedding $\mathcal{A} \to \mathcal{A}^{perf}$ induces an isomorphism $HH_*(\mathcal{A}) \xrightarrow{\simeq} HH_*(\mathcal{A}^{perf})$.

   PROOF: $\mathcal{A} \to \mathcal{A}^{perf}$ induces a restriction map $\mathcal{A}^{perf}$-bimodules$\to$ $\mathcal{A}$-bimodules, which is a quasi-equivalence sending the diagonal to the diagonal ▶

4. Künneth formula: $HH_*(\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}) \cong HH_*(\mathcal{A}) \otimes HH_*(\mathcal{B})$.

   To see this, use Eq. 1 and pass to d.g. categories.

5. Opposite property: $HH_*(\mathcal{A}^{pop}) \cong HH_*(\mathcal{A})$.

   This follows from the 'concrete' definition.

6. Normalization: $HH_*(\mathbb{K}) = \begin{cases} \mathbb{K} & * = 0 \\ 0 & else \end{cases}$

   Moreover, $\forall$ $P \in Ob(\mathbb{K}^{perf})$, there is a map $\mathbb{K} \to \mathbb{K}^{perf}$. This induces a map

   $$\mathbb{K} \cong HH_*(\mathbb{K}) \to HH_*(\mathbb{K}^{perf}) \overset{Morita}{\cong} HH_0(\mathbb{K}) \cong \mathbb{K}.$$

   This map is multiplication by χ.

7. Exactness: Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be a full A_∞ subcategory and $\mathcal{A}/\mathcal{B}$ the quotient. Then there exists a long exact sequence

   $$... \to HH_*(\mathcal{B}) \to HH_*(\mathcal{A}) \to HH_*(\mathcal{A}/\mathcal{B}) \to HH_{*+1}(\mathcal{B}) \to ...$$

8. (unproperty) HH_* does <u>not</u> satisfy homotopy invariance, which is when the inclusion $\mathbb{K} \to \mathbb{K}[t]$ induces an isomorphism in HH_*. It is doubtful that there exists a homology theory with all of the above properties.

3.4 Consequences of the key properties

1. Let $\mathbb{K} \to \mathcal{A}^{perf}$ be the functor associated to $P \in Ob(\mathcal{A}^{perf})$. Then the image of 1 under

   $$\mathbb{K} \overset{norm}{=\!=} HH_0(\mathbb{K}) \to HH_0(\mathcal{A}^{perf}) \overset{Morita}{\cong} HH_0(\mathcal{A})$$

   is denoted $[P]_{HH}$.

   By functoriality, this is invariant of the quasi-isomorphism class of P. By normalization,

   $$[P[k]]_{HH} = (-1)^k [P]_{HH}.$$

2. Let $M \in Ob(\mathcal{A}^{prop})$. M is (by definition) a functor $\mathcal{A}^{opp} \to \mathbb{K}^{prop} \cong \mathbb{K}^{perf}$. We can then consider

   $$HH_0(\mathcal{A}) \overset{opp}{=\!=} HH_0(\mathcal{A}^{opp}) \longrightarrow HH_0(\mathbb{K}^{prop}) \cong HH_0(\mathbb{K}^{perf}) \cong HH_0(\mathbb{K}) \cong \mathbb{K}.$$

   This associates to M the class $[M]^\vee_{HH} \in HH_0(\mathcal{A})^\vee$.

3. Recall that if P is a perfect module and M is a proper module, then $\mathsf{H}^*(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{P},\mathsf{M}))$ is finite dimensional (i.e.. $\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{P},\mathsf{M}) \in \mathrm{Ob}(\mathbb{K}^{\mathrm{prop}})$).

   Let $\mathcal{F}_{\mathsf{P}} : \mathbb{K} \to \mathcal{A}^{\mathrm{perf}}$ be the obvious functor, and similarly for $\mathcal{F}_{\mathsf{M}} : \mathcal{A}^{\mathrm{opp}} \to \mathbb{K}^{\mathrm{perf}}$. Also define

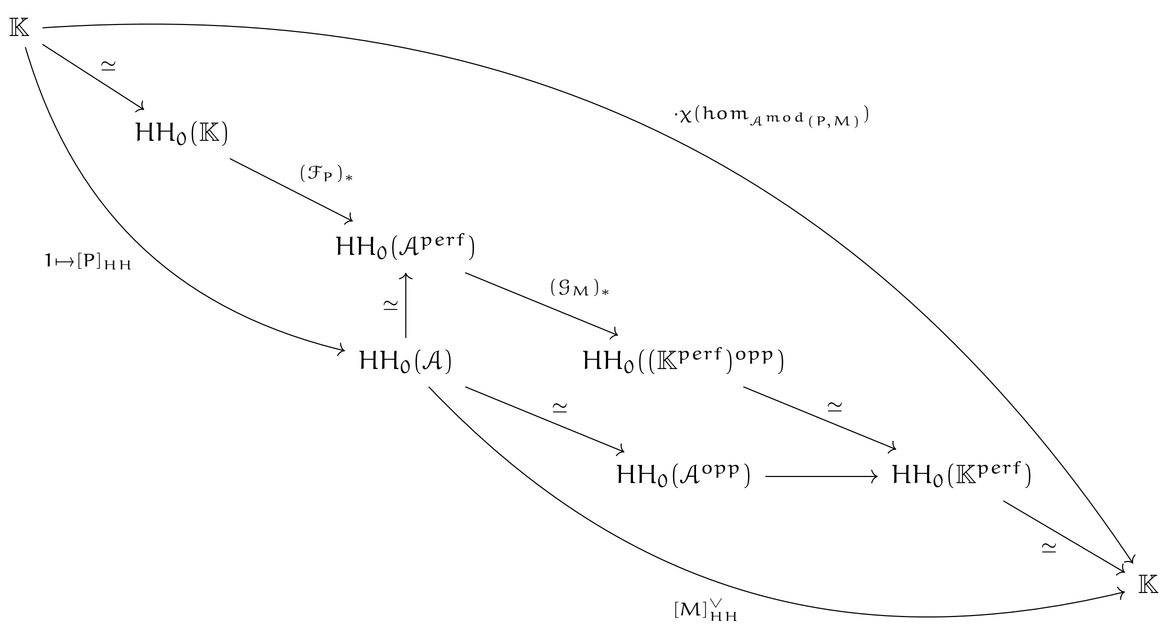
$$\begin{aligned} \mathcal{G}_{\mathsf{M}} : \mathcal{A}^{\mathrm{perf}} &\to (\mathbb{K}^{\mathrm{prop}})^{\mathrm{opp}} \\ \mathsf{Q} &\mapsto \mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{Q},\mathsf{M})). \end{aligned}$$

   Then $\mathcal{G}_{\mathsf{M}} \circ \mathcal{F}_{\mathsf{P}} : \mathbb{K} \to (\mathbb{K}^{\mathrm{prop}})^{\mathrm{opp}}$ induces a map

$$\mathsf{HH}_0(\mathbb{K}) \longrightarrow \mathsf{HH}_0((\mathbb{K}^{\mathrm{prop}})^{\mathrm{opp}}) \longrightarrow \mathsf{HH}_0(\mathbb{K}),$$

   and this map is precisely multiplication by $\chi(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{P},\mathsf{M}))$. We obtain the "Cardy relation"

$$\langle [\mathsf{M}]^{\vee}_{\mathsf{HH}}, [\mathsf{P}]_{\mathsf{HH}} \rangle = \chi\left(\mathsf{H}^*(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{P},\mathsf{M}))\right). \tag{2}$$



4 Lecture 3: Hochschild homology contd. and Hochschild cohomology

4. There is a natural A_∞-functor

$$\begin{aligned} \mathcal{A}^{\mathrm{prop}} \otimes (\mathcal{A}^{\mathrm{perf}})^{\mathrm{opp}} &\longrightarrow \mathbb{K}^{\mathrm{prop}} \\ (\mathsf{M},\mathsf{P}) &\mapsto \mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{P},\mathsf{M}) \end{aligned}$$

   inducing

$$\begin{aligned} \mathsf{HH}_*(\mathcal{A}^{\mathrm{prop}}) \otimes \mathsf{HH}_*(\mathcal{A}^{\mathrm{perf}}) \cong \mathsf{HH}_*(\mathcal{A}^{\mathrm{prop}} \otimes (\mathcal{A}^{\mathrm{perf}})^{\mathrm{opp}}) &\longrightarrow \mathsf{HH}_*(\mathbb{K}^{\mathrm{prop}}) \cong \mathbb{K} \\ [\mathsf{M}] \otimes [\mathsf{P}] &\mapsto \chi(\mathsf{H}^*(\mathrm{hom}_{\mathcal{A}^{\mathrm{mod}}}(\mathsf{P},\mathsf{M}))). \end{aligned}$$

Recall: If Q is a perfect $(\mathcal{A}, \mathcal{B})$-bimodule, then the convolution functor

$$\Phi_Q : \mathcal{A}^{mod} \longrightarrow \mathcal{B}^{mod}$$
$$M \mapsto M \otimes_{\mathcal{A}} Q$$

brings proper modules to perfect modules.

5. Q as above, so $[Q]_{HH} \in HH_*(\mathcal{A}^{pop} \otimes \mathcal{B}) \cong HH_*(\mathcal{A}) \otimes HH_*(\mathcal{A}) \otimes HH_*(\mathcal{B})$. The induced map

$$(\Phi_Q)_* : HH_*(\mathcal{A}^{mod}) \to HH_*(\mathcal{B}^{mod})$$

is given by contraction with $[Q]_{HH}$.

Recall: $\mathcal{A}$ is a smooth A_∞ category if the diagonal bimodule is perfect.

6. If $\mathcal{A}$ is smooth, then the map $(\Phi_{\mathcal{A}})_* : HH_*(\mathcal{A}^{prop}) \to HH_*(\mathcal{A}^{perf}) \cong HH_*(\mathcal{A})$ has finite rank (since $\Phi_{\mathcal{A}} \cong Id_{\mathcal{A}}$). So we can think of $(\Phi_{\mathcal{A}})_*$ as induced by the inclusion $\mathcal{A}^{prop} \to \mathcal{A}^{perf}$.

Recall: If $\mathcal{A}$ is proper then $\mathcal{A}^{perf} \subset \mathcal{A}^{prop}$.

7. $\mathcal{A}$ proper $\implies$ we can restrict eq. (2) to get a pairing

$$(\cdot, \cdot)_{HH} : HH_*(\mathcal{A}) \otimes HH_*(\mathcal{A}) \longrightarrow \mathbb{K}.$$

8. So if $\mathcal{A}$ is smooth and proper, then $HH_*(\mathcal{A})$ has finite total dimension. Moreover, the above pairing is non-degenerate.

9. If a homology theory $\mathbb{H}$ satisfies exactness, then it also satisfies weak exactness: If $\mathcal{A}$ is a directed A_∞ category with objects $X_1, ..., X_n$, then

$$\mathbb{H}_*(\mathcal{A}) \cong \begin{cases} \mathbb{K}^m & * = 0 \\ 0 & else \end{cases}$$

10. The Grothendieck group is

$$K_0(\mathcal{A}) = \langle [C]_K ; C \in Ob(\mathcal{A}) \rangle / \{[Cone(C_1 \to C_2)]_K = [C_2]_K - [C_1]_K\}.$$

If $\mathbb{H}_*$ satisfies weak exactness then $[P]_{\mathbb{H}}$ defines a group homomorphism

$$K_0(\mathcal{A}^{perf}) \to \mathbb{H}_0(\mathcal{A}),$$

$[M]_{\mathbb{H}}$ defines a group homomorphism

$$K_0(\mathcal{A}^{prop}) \to \mathbb{H}_0(\mathcal{A})^\vee,$$

and $\mathbb{H}_*(\Phi_Q)$ depends only on $[Q]_K \in K_0((\mathcal{A}, \mathcal{B})^{perf})$.

11. HH_* does not satisfy homotopy invariance; instead, it satisfies

$$HH_*(\mathbb{K}[s]) \cong \begin{cases} \mathbb{K}[s] & * = 0, -1 \\ 0 & else \end{cases} .$$

This allows for a "Lefschetz trace" type formula.

EXAMPLE 6) Let C be a chain complex and c an endomorphism, which induces the A_∞-functor

$$\begin{aligned} \mathbb{K}[s] &\longrightarrow \mathbb{K}^{perf} \\ s &\mapsto c. \end{aligned}$$

The induced map is

$$\begin{aligned} \mathbb{K}[s] \cong HH_0(\mathbb{K}[s]) &\longrightarrow HH_0(\mathbb{K}^{perf}) \cong \mathbb{K} \\ s^k &\overset{(*)}{\longmapsto} Str(c^k). \end{aligned}$$

(*) can be written $\sum_k u^k Str(c^k)$, where u^k is dual to s^k. Likewise, given $P \in \mathcal{A}^{perf}$ with endomorphism p, $[p]_{HH} \in HH_0(\mathcal{A})[[u]]$. Then for $M \in Ob(\mathcal{A}^{prop})$,

$$\langle [M]^\vee_{HH}, [p]_{HH} \rangle = \sum_k u^k Str(\cdot [p]^k : H^k(hom_{\mathcal{A}^{mod}}(P, M) \to H^k(hom_{\mathcal{A}^{mod}}(P, M)))).$$

5 Hochschild Cohomology

5.1 Motivation

1. HH^* has the structure of both a graded Lie algebra and a graded commutative algebra, thus it has a Gerstenhaber algebra structure.
2. HH^* encodes information about deformations of A_∞ structures.

5.2 Definitions

Assuption: Char $\mathbb{K} = 0$. [Not strictly necessary, but some results here no longer hold without it.]

DEFINITION 13 Let $\mathcal{A}$ be a graded vector space. Define

$$T(\mathcal{A}[1]) = \mathbb{K} \oplus \mathcal{A}[1] \oplus (\mathcal{A} \otimes \mathcal{A})[2] \oplus ...$$

$T(\mathcal{A}[1])$ is a coalgebra with coproduct

$$a_d \otimes ... \otimes a_1 \mapsto \sum_{i=0}^{d} (a_d \otimes ... \otimes a_{i+1}) \otimes (a_i \otimes ... \otimes a_1).$$

DEFINITION 14 A coderivation is a map Γ satisfying the coLeibnitz rule

$$\Delta \circ \Gamma = (\Gamma \otimes id + id \otimes \Gamma) \circ \Delta.$$

DEFINITION 15

$$CC^*(\mathcal{A}, \mathcal{A}) := Hom(T(\mathcal{A}[1]), \mathcal{A}) = \prod_{d \geq 0} Hom(\mathcal{A}^{\otimes d}, \mathcal{A})[-d].$$

LEMMA 5 $Hom(T(\mathcal{A}[1]), \mathcal{A}) \cong Coder(T(\mathcal{A}[1]))[-1]$.

PROOF: $(\longleftarrow)$ Compose coderivation with the projection onto $\mathcal{A}$.

$(\longrightarrow)$ $\gamma \mapsto Coder\Gamma$, where $\Gamma(a_d \otimes ... \otimes a_1) = \sum_{i,j} a_d \otimes ... \otimes \gamma(a_{i+j} \otimes ... \otimes a_{i+1}) \otimes a_i \otimes ... \otimes a_1$. ▶

5.3 Graded Lie Algebra Structure

$[\gamma_1, \gamma_2](a_d \otimes ... \otimes a_1)$" $=$" $\gamma_1 \circ \gamma_2 \pm \gamma_2 \circ \gamma_1$.

Given $\gamma \in CC(\mathcal{A}, \mathcal{A})$, we want to define $\exp(\gamma) : T(\mathcal{A}[1]) \to T(\mathcal{A}[1])$.

Compose with projection to get $\mathcal{F} : T(\mathcal{A}[1]) \to \mathcal{A}[1]$.

Has components $\mathcal{F}^0 \in \mathcal{A}$, $\mathcal{F}^1 : \mathcal{A} \to \mathcal{A}$, $\mathcal{F}^2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[-1]$.

"Define" $\exp(\gamma) = \mathrm{Id} + \Gamma + \frac{1}{2}\Gamma^2 + \frac{1}{3!}\Gamma^3 + ...$

$$\mathcal{F}^0 = \gamma^0 + \frac{1}{2}\gamma^1(\gamma^0) + 2\frac{1}{6}\gamma^2(\gamma^0, \gamma^0) + \frac{1}{6}\gamma^1(\gamma^1(\gamma^0)) + ...$$

$$\mathcal{F}^1 = \mathrm{id}_{\mathcal{A}} + \gamma^1 + \frac{1}{2}\gamma^1(\gamma^1) + \frac{1}{2}\gamma^2(\cdot, \gamma^0) + \frac{1}{2}\gamma^2(\gamma^), \cdot) + ...$$

5.4 Problem of infinite sums

Solution 1: $CC^*(\mathcal{A}, \mathcal{A})$ has a decreasing filtration

$$F^p CC^*(\mathcal{A}, \mathcal{A}) = \prod_{d \geq p} \mathrm{Hom}(\mathcal{A}[1]^{\otimes d}, \mathcal{A}).$$

Easy to check that the Lie bracket interacts as

$$[F^p CC^*, F^q CC^*] = F^{p+q-1} CC^*.$$

Assume $\gamma \in F^2 CC^*$ (i.e. $\gamma^0 = \gamma^1 = 0$). Then $\exp(\gamma)$ is well-defined. This is true more generally if γ^1 is nilpotent.

Solution 2: Introduce formal variable T to make exp converge in $\mathbb{K}[T]$.

5.5 Deformations of A_∞ structures

Now assume $\mathcal{A}$ carries an A_∞ structures $\mu_{\mathcal{A}}$. A non-unital A_∞ structure μ can be throughout of as an element in CC^2 such that $\mu^0 = 0$ and $\frac{1}{2}[\mu, \mu] = 0$.

5.5.1 First order deformations

Introduce a formal variable ϵ. Suppose we have a first order deformation $\mu_\epsilon := \mu + \epsilon\gamma$, where $\gamma \in F^1 CC^2(\mathcal{A}, \mathcal{A})$ and $\epsilon^2 = 0$. Then

$$\frac{1}{2}[\mu_\epsilon, \mu_\epsilon] = \epsilon[\mu, \gamma].$$

DEFINITION 16 The differential on $CC^*(\mathcal{A})$ is $\partial = [\mu, \cdot]$. Its cohomology is HH^*.

First order deformations $\iff$ degree 2 Hochschild cocycles.

Remark: $[,]$ on CC^* induces a graded Lie algebra structure on HH^*.

5.5.2 Second order deformations

$\mu_\epsilon = \mu + \epsilon\gamma + \epsilon^2\delta$, where $\epsilon^3 = 0$ and γ is a cocycle.

$$[\mu_\epsilon, \mu_\epsilon] = \epsilon^2(\partial\delta + [\gamma, \gamma]).$$

A first order deformation can be extended to a second order deformation iff $[[\gamma], [\gamma]]$ vanishes.

THEOREM 2 *Suppose $\mathcal{A}$ is a graded algebra such that $HH^*(\mathcal{A}, \mathcal{A}[2-p]) = 0$ for all $p \geq 3$. Then any A_∞-algebra $\mathbf{A}$ such that $H^*(\mathbf{A}) \cong \mathcal{A}$ is quasi-isomorphic to $\mathcal{A}$ as an A_∞ algebra.*

PROOF: Consider A_∞ deformations of $\mathcal{A}$. $H(\mathbf{A}) \cong \mathcal{A} \implies$ no first order deformations. Assumptions mean they also agree in higher order. ▶

Remark: $\mathcal{A}$ a graded algebra $\implies$ can define $HH^*(\mathcal{A}, \mathcal{A}[q])$.

5.6 Product structure

$HH^*(\mathcal{A}, \mathcal{A})$ has a graded product structure induced by

$$\gamma_2 * \gamma_1 = \sum \pm\mu(\ldots, \gamma_2(\ldots), \ldots, \gamma_1(\ldots), \ldots).$$

LEMMA 6 *The induced product on cohomology is graded commutative.*

THEOREM 3 *Any first order deformation of the diagonal bimodule $\mathcal{A}$ can be extended to arbitrarily high order (here we need $\mathrm{char}\mathbb{K} = 0$).*

PROOF: Let $[\gamma] \in HH^1(\mathcal{A}, \mathcal{A})$ and $\mathcal{F} = \exp(t\gamma)$ where t is a formal variable. Take graph biomdule of $\mathcal{F}$. $\implies$ deformation over $\mathbb{K}[[t]]$. ▶