Self-adjoint maps :

Def. An operator is called self-adjoint if $T: V \to V$ satisfies $T = T^*$.

In other words $\underbrace{\langle Tv, w\rangle = \langle v, Tw\rangle}$ $\forall\, v, w \in V$.

Ex. $V = \mathbb{R}^2$, T given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ w.r.t. the standard basis.

$$T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \boxed{ax_1x_2 + by_1x_2 + cx_1y_2 + dy_1y_2}.$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{bmatrix} = \boxed{ax_1x_2 + bx_1y_2 + cy_1x_2 + dy_1y_2}.$$

$T = T^*$ precisely when $b = c$.

i.e. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is symmetric.

$$Ax \cdot y = x \cdot A^T y.$$

A self-adjoint $\iff$ $A = A^T$

$\iff$ A symmetric.

<u>Prop</u>. Eigenvalues of a self-adjoint operator are real.

Pf. If λ is an eigenvalue of a self-adjoint T then:

$$\lambda \|v\|^2 = \langle \lambda v, v \rangle$$

$$= \langle Tv, v \rangle$$ if v is an eigenvector of λ

$$= \langle v, Tv \rangle$$

$$= \langle v, \lambda v \rangle$$

$$= \bar{\lambda} \langle v, v \rangle$$

$$= \bar{\lambda} \|v\|^2.$$

$\Rightarrow \lambda = \bar{\lambda}$. in other words, $\lambda \in \mathbb{R}$.

Note: The proof, while straight-forward relies on a very fundamental characterization of $\mathbb{R}$ as the subspace of $\mathbb{C}$ fixed under complex conjugation.

Cor. Eigenvalues of symmetric matrices are real.

In particular, the covariance matrix in higher dimensions has real eigenvalues.

$\boxed{\text{Prop. If } T = T^* \text{ then } \exists \text{ an orthonormal basis of eigenvectors.}}$

Gives us: projection onto components.

Principal Component Analysis.

Pf. $\boxed{\text{In case: } F = \mathbb{C}.}$

We know T has an eigenvalue λ.

only place in pf. where we used $F = \mathbb{C}$.

Let e_1 be an eigenvector of λ.

Let $\mathcal{U} = \operatorname{span}(e_1)^{\perp}$

$u \in \mathcal{U}$

So: if we show that self-adjoint operators on V over $\mathbb{R}$ have an eigenvalue, the proof goes through.

Because T is self-adjoint,

$$\begin{aligned}\langle Tu, e_1\rangle &= \langle u, Te_1\rangle \\ &= \langle u, \lambda e_1\rangle \\ &= \bar{\lambda}\langle u, e_1\rangle \\ &= 0.\end{aligned}$$

$\rightsquigarrow \mathcal{U}$ is an invariant subspace of T.

$T|_{\mathcal{U}}$ defines a self-adjoint operator on $\mathcal{U}$.

$T|_{\mathcal{U}}$ has an eigenvalue.

Repeat process...

To make precise, use induction on dimension.

Base case. If $\dim(V) = 1$ we are done.

$\boxed{\text{Suppose the proposition holds whenever } \dim(V) \leq k.}$

Suppose $\dim(V) = k+1$

T has an eigenvalue λ with eigenvector e_1.

$\mathcal{U} = \operatorname{span}(e_1)^{\perp}$ is invariant under T.

By hypothesis, since $\dim(\mathcal{U}) = k$, $\mathcal{U}$ has an orthonormal basis of eigenvectors $\{u_1, \dots, u_k\}$.

Then $\{u_1, \dots, u_k, \frac{e_1}{\|e_1\|}\}$ is an orthonormal basis for V.

$\blacksquare$

Prop. If $F = \mathbb{R}$, then $T: V \to V$ admits an orthonormal basis of eigenvectors $\iff$ T is self-adjoint.

If $F = \mathbb{C}$ then $T: V \to V$ admits an orthonormal basis of eigenvectors $\iff$ T is <u>normal</u>: $\underbrace{TT^* = T^*T}_{T \text{ commutes with } T^*}$.

<u>e.g.</u> (normal operator that is not self-adjoint)

T given by matrix $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = A$

T^* is given by $A^\dagger = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$ (so $A \neq A^\dagger$)

However TT^* is given by $AA^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^\dagger A$.

<u>Prop.</u> T is normal $\iff \|Tv\| = \|T^*v\| \quad \forall\, v$.

<u>Pf.</u> $TT^* - T^*T = 0$

$\iff \langle (TT^* - T^*T)v, v \rangle = 0$.

$\iff \langle TT^*v, v \rangle = \langle T^*Tv, v \rangle$.

$\iff \langle T^*v, T^*v \rangle = \langle Tv, Tv \rangle$.

$\Longleftrightarrow \quad \|T^*v\|^2 = \|Tv\|^2.$

$\square$

Prop. Eigenvectors of a normal operator T corresponding to distinct eigenvalues are orthogonal.

Pf. $Tu = \alpha u$, $Tv = \beta v$, $\alpha, \beta \in F$ and $\alpha \neq \beta$.

$\underbrace{\text{W.T.S.}}_{\text{want to show}}$ that $\langle u, v\rangle = 0$.

$$(\alpha-\beta)\langle u, v\rangle = \alpha\langle u, v\rangle - \beta\langle u, v\rangle$$
$$= \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle$$
$$= \langle Tu, v\rangle - \langle u, \bar{\beta} v\rangle$$

If we can show that $\bar{\beta} v = T^* v$, we are done.

Because then, $(\alpha - \beta)\langle u, v\rangle = \langle Tu, v\rangle - \langle u, T^* v\rangle$

$$= \langle Tu, v\rangle - \langle Tu, v\rangle$$

$$= 0.$$

Since $\alpha \neq \beta$, $\alpha - \beta \neq 0$

$\Rightarrow \langle u, v\rangle = 0$.

Exercise: no additional info. is needed for T self-adjoint and $F = \mathbb{R}$.