"No one gets excited about vector spaces"

Objectives for today

① Give more examples of subspaces

② Define span and direct sum

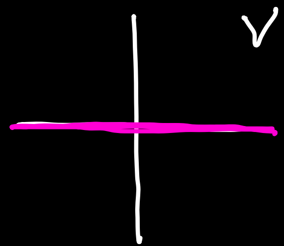
③ Prove that span is the smallest containing subspace.

④ Give conditions for a sum of subspaces to be a direct sum.

SUBSPACES

<u>Definition</u>. A <u>subspace</u> $\mathcal{U}$ of a vector space $\mathcal{V}$ is a subset of $\mathcal{V}$ that is a vector space with operations $+, \cdot$ induced by $+, \cdot$ on $\mathcal{V}$.

<u>Ex 1</u>. $U = \{(x, 0) \mid x \in \mathbb{R}\}$, $V = \mathbb{R}^2$.



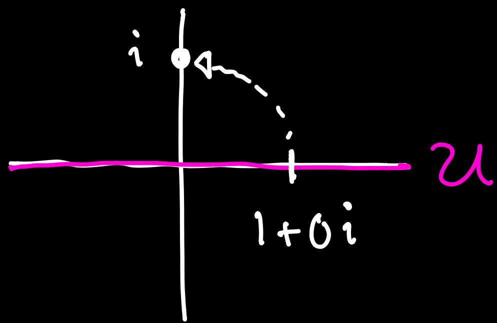
<u>Ex 2</u>. $U = \{(x, x, x) \mid x \in \mathbb{R}\}$, $V = \mathbb{R}^3$

<u>Ex 3</u>. $U = \{\vec{0}\}$, $V = \mathbb{R}^n$.

<u>Ex 4</u>. A plane in $\mathbb{R}^3$ ...

non-example: $\mathcal{U} = \{x + 0i\} \subseteq \mathbb{C}$, viewed as a vector space over $\mathbb{C}$.

$\mathcal{U}$ is not a subspace.



$$i \cdot (1 + 0i) =$$
$$i + 0\, i \cdot i =$$
$$i \notin \mathcal{U}.$$

So $\mathcal{U}$ is not closed under scalar multiplication.

Proposition $\mathcal{U} \subseteq V$ is a subspace of V

the set $\mathcal{U}$ is contained in the set V

if and only if the following three conditions hold:

(i) $\vec{0} \in \mathcal{U}$ (the additive identity of V is in $\mathcal{U}$)

(ii) $\forall\, u, v \in \mathcal{U}$ $u + v \in \mathcal{U}$ ($\mathcal{U}$ is closed under addition)

(ii) $\forall\, u \in \mathcal{U}$ and $c \in \mathbb{F}$

$c \cdot u \in \mathcal{U}$. ($\mathcal{U}$ is closed under scalar multiplication)

Proof. Since $\mathcal{U}$ is a subspace, $\vec{0} \in \mathcal{U}$.

$(\Rightarrow)$ Suppose $\mathcal{U}$ is a subspace

Similarly, $\mathcal{U}$ is closed under vector addition and scalar multiplication.

$(\Leftarrow)$ Suppose $\mathcal{U}$ satisfies the 3 conditions.

Since $\mathcal{U} \subseteq V$, $x \in \mathcal{U} \Rightarrow x \in V$. Thus, Properties 1, 2, and 5-8 hold because V is a vector space. By assumption, addition and scalar multiplication are well-defined. So it remains to show

closure of addition + scalar multiplication

that Properties 3 and 4 hold.

(3) Let $\vec{0} \in V$ be the additive identity. Let $u \in \mathcal{U}$. Then $u \in V$ and $\vec{0} + u = u$. Thus, $\vec{0}$ is also the additive identity for $\mathcal{U}$.

(4) Let $u \in \mathcal{U}$. Then $u \in V$ and so u has an additive

inverse $-u \in V$. By Prop. 1.31 in Axler,

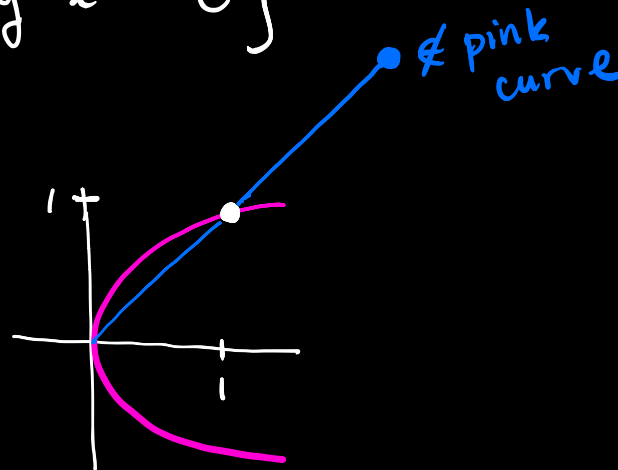
($-v = (-1)\cdot v$)

$-u = (-1)\cdot u$.

By assumption, $(-1)\cdot u \in U$.

Thus, u has an additive inverse in U. $\square$

<u>Non-ex</u>. $U = \{ (x,y) \in \mathbb{R}^2 \mid y - x^2 = 0 \}$



U is <u>not</u> a subspace.

It suffices to show that U is not closed under scalar multiplication.

$(1,1) \in U$. Is $2\cdot(1,1) = (2,2) \in U$? No: $2 - 4 \neq 0$.

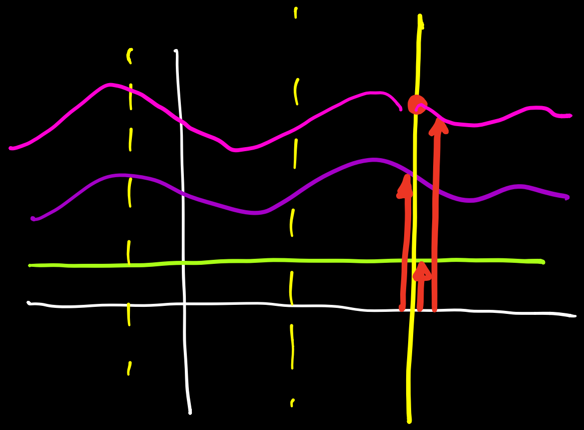
$\underline{Ex}$. $\mathcal{F}(\mathbb{R}) = \{\text{functions from } \mathbb{R} \to \mathbb{R}\}$ is a vector space.

$\mathcal{U} = \{\text{continuous functions from } \mathbb{R} \to \mathbb{R}\}$ is a subspace.

(i) $f = 0$ is cont.

(ii) sums of cont. functions are continuous.

(iii) $f \in \mathcal{U}$, $c \in \mathbb{R}$, then $c \cdot f$ is continuous.



Ex $\mathcal{P}_n(\mathbb{R}) = \{$ polynomials over $\mathbb{R}$ of degree $\leq n\}$ is a subspace of $\mathcal{P}(\mathbb{R}) = \{$all polynomials over $\mathbb{R}\}$.

Ex. $F^{\infty} = \{(x_1, x_2, x_3, \ldots) \mid x_i \in F\}$ is a vector space.

$$\mathcal{U} = \{(u_1, u_2, u_3, \ldots) \mid u_i \in F,\ u_n = u_{n-1} + u_{n-2},\ \text{when } n \geq 2\}$$

is a subspace.

FIBONACCI SEQUENCE

Def. Let V be a vector space, and let S be a list of elements of V.

① A <u>linear combination</u> of elements in S is any sum of the form $a_1 v_1 + a_2 v_2 + \dots + a_m v_m$, $a_i \in F$, $v_i \in S$

ⓑ If S is non-empty, the (linear) span of S is

$$\text{Span}(S) = \{\text{linear combinations of elements in } S\}$$

If S is empty, $S = \emptyset$, $\text{Span}(S) = \{0\}$.

ⓒ If $W = \text{Span}(S)$ then S <u>spans</u> or <u>generates</u> W.

<u>Ex</u>. $V = \mathbb{R}^3$, $S = \{(1,0,0), (0,1,0)\}$, then

$$\text{Span}(S) = \{(a, b, 0) \mid a, b \in \mathbb{R}\}.$$

Prop. Span(S) is the smallest subspace containing S.

Pf. Let $S = \{v_1, \dots, v_m\}$.

$\vec{0} \in \text{Span}(S)$ because $\vec{0} = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_m$.

Span(S) is closed under addition because

$$(a_1 v_1 + \dots + a_m v_m) + (b_1 v_1 + \dots + b_m v_m) =$$

$$(a_1 + b_1) v_1 + \dots + (a_m + b_m) v_m .$$

Span(S) is closed under scalar multiplication because $c \cdot (a_1 v_1 + \dots + a_m v_m) = c \cdot a_1 v_1 + \dots + c \cdot a_m v_m$.

By the Proposition, Span(S) is a subspace.

$S \subseteq \text{Span}(S)$ because

$$v_i = 0 \cdot v_1 + \dots + 0 \cdot v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_m .$$

If W is a subspace, $W \supset S$, then W is closed under addition and scalar multiplication, and so $W \supset \mathrm{Span}(S)$.

<u>BTW</u>. If you want to show that two sets U, V are equal. One way:

$$U \subset V \quad \text{and}$$

$$V \subset U .$$

Definition Let $\mathcal{U}_1, \ldots, \mathcal{U}_m$ be subspaces of a vector space V. The **sum**, denoted by $\mathcal{U}_1 + \ldots + \mathcal{U}_m$ is the set

$$\{ u_1 + \ldots + u_m \mid u_i \in \mathcal{U}_i \}$$

Definition The sum $\mathcal{U}_1 + \ldots + \mathcal{U}_m$ is a **direct sum** written $\mathcal{U}_1 \oplus \ldots \oplus \mathcal{U}_m$ if each element in $\mathcal{U}_1 + \ldots + \mathcal{U}_m$ can be written **uniquley** as a sum $u_1 + \ldots + u_m$, where $u_i \in \mathcal{U}_i$.

Ex $\text{Span}\{(1,0,0)\} + \text{Span}\{(0,1,0)\}$ in $\mathbb{R}^3$ is a direct sum.

<u>Ex</u>. $\mathcal{U}_1 = \{ax\}$, $\mathcal{U}_2 = \{bx^2+bx\}$ $a, b, c \in \mathbb{R}$

$\mathcal{U}_1, \mathcal{U}_2$ subsets of $\mathcal{P}(\mathbb{R})$.

<u>Claim</u> $\mathcal{U}_1 + \mathcal{U}_2$ is a direct sum.

How do we see this?

$$0 = ax + bx^2 + ax$$

$$= (a+b) \cdot x + bx^2.$$

$\Rightarrow b = 0$, $a + b = 0$. $\Rightarrow a = 0$.

Prop. $\mathcal{U}_1 + \dots + \mathcal{U}_m$ is a <u>direct sum</u> $\iff$

$\nexists$ $u_1, \dots, u_m$ where $u_i \in \mathcal{U}_i$ and some $u_i \neq 0$

and $\vec{0} = u_1 + \dots + u_m$.

Pf.

($\Rightarrow$) Suppose $\mathcal{U}_1 + \dots + \mathcal{U}_m$ is a direct sum. By definition, $\vec{0}$ is the unique sum $\vec{0} + \dots + \vec{0}$.

($\Leftarrow$) Write $\vec{v} = v_1 + \dots + v_m$, $v_i \in \mathcal{U}_i$

$\vec{v} = w_1 + \dots + w_m$, $w_i \in \mathcal{U}_i$.

Then $\vec{0} = \vec{v} - \vec{v} = (v_1 + \dots + v_m) - (w_1 + \dots + w_m)$

$= (v_1 - w_1) + \dots + (v_m - w_m)$, where

$v_i - w_i \in \mathcal{U}_i$.

By assumption, $v_i - w_i = \vec{0}$.

$$v_i = w_i.$$

$\square$

<u>Exercise</u>

Is $\mathcal{U}+\mathcal{V}$ a subspace?

Is $\mathcal{U} \cup \mathcal{V} = \{ u \mid u \in \mathcal{U} \text{ or } u \in \mathcal{V} \}$ a subspace?

Is $\mathcal{U} \cap \mathcal{V} = \{ u \mid u \in \mathcal{U} \text{ and } u \in \mathcal{V} \}$ a subspace?

(1.45) $\mathcal{U}+\mathcal{V}$ is a <u>direct sum</u> $\iff$ $\mathcal{U} \cap \mathcal{V} = \{0\}$.