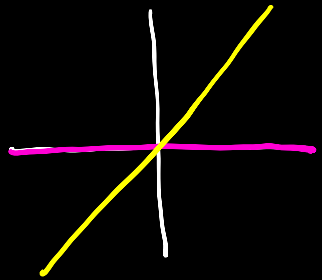
TAKING STOCK

1. Axiomatic definition of a <u>field</u>.
   - Think $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$
   - Think $\mathbb{F}_p$ to test my intuition.
2. Axiomatic definition of a <u>vector space</u>.
   - Intuition: "everything works as nicely as possible"
   - $\boxed{\text{Think: } \mathbb{R}^n, \mathbb{C}^n, \mathbb{Q}^n.}$
   - Think $P_n(\mathbb{R})$, $P(\mathbb{R})$, $\mathcal{F}(\mathbb{R})$ for more complexity

3. Subspaces: "vector spaces inside of vector spaces".

- Think $\mathbb{R} \subseteq \mathbb{R}^n$



- Think $\mathbb{R}^m \subseteq \mathbb{R}^n$ $m \leq n$.

- $P_m(\mathbb{R}) \subseteq P_n(\mathbb{R})$

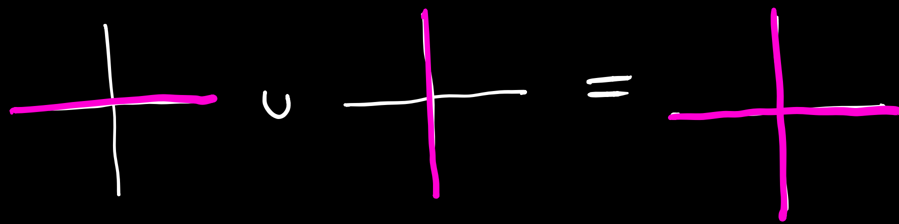
- $C^0(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$

PROP: $\mathcal{U} \subseteq V$ is a subspace $\iff \vec{0} \in \mathcal{U}$, and $\mathcal{U}$ is closed under vector addition and scalar multiplication.

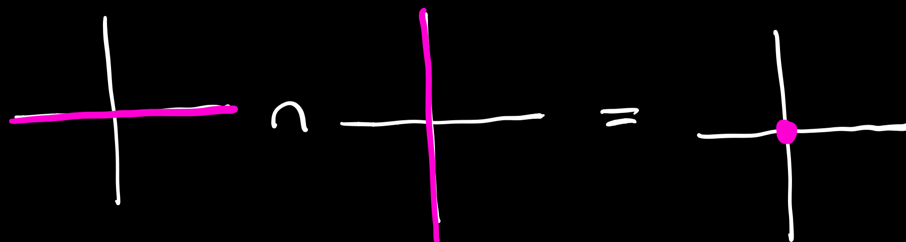
MORAL: ALWAYS CHECK THAT THINGS ARE WELL-DEFINED

4. Operations on Subspaces

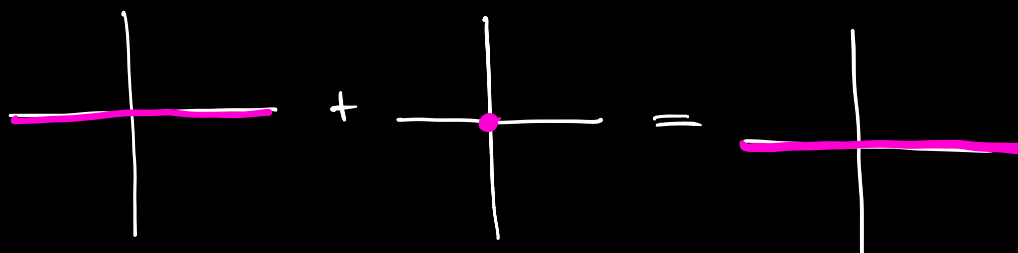
- union



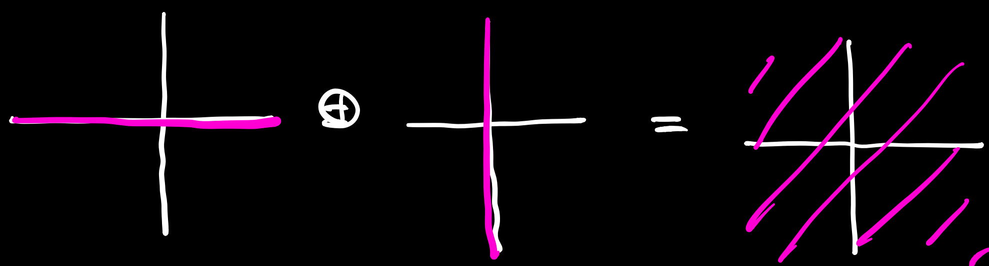
- intersection



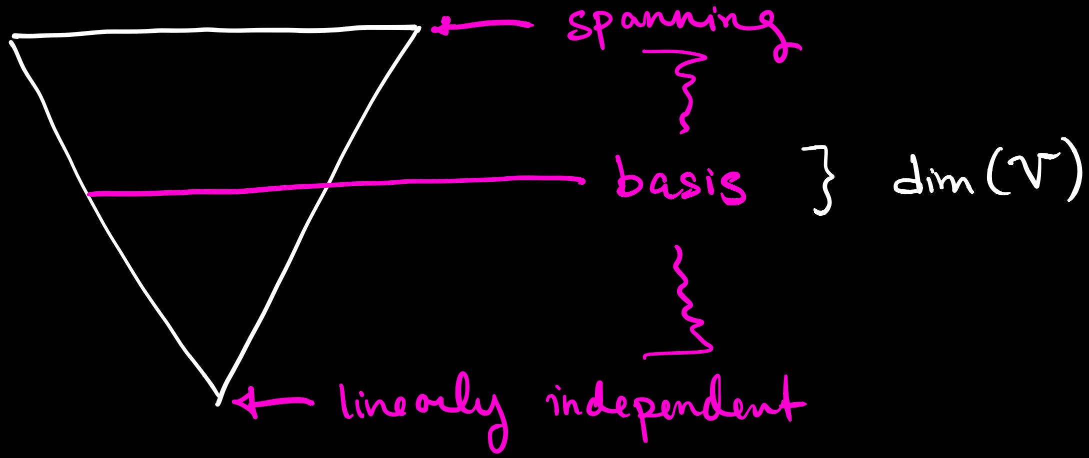
- sum



- direct sum



5. BASES & DIMENSION



A linearly independent list of size dim(V) is a basis.

A spanning list of size dim(V) is a basis

- Think $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ basis of $\mathbb{R}^n$.

- Think $\{1, x, x^2, \ldots, x^n\}$ basis of $\mathcal{P}_n(\mathbb{R})$

PROP All finite-dimensional vector spaces have a basis, and the size of any two bases is the same.

$\Rightarrow$ Allows us to define dimension.

6. Methods of proofs

(i) Contradiction : $u \in \mathcal{U}$, $v \in V \setminus \mathcal{U}$. Is $u+v \in \mathcal{U}$?

Assume for contradiction that $\exists\, w \in \mathcal{U}$, and $u+v = w$.

Then $v = (-u) + w \in \mathcal{U}$ b/c $\mathcal{U}$ is a subspace, contradicting $v \in V \setminus \mathcal{U}$.

(ii) Counter-example : $\mathcal{U} \neq W$ s.t. $\mathcal{U} + V = W + V$.

$V = \mathcal{U} = \mathbb{R}$, $W = \{0\}$.

(iii) Induction . . .

To Be Contined.

RECALL V vector space over F with a basis $\{v_1, \dots, v_n\}$, there is a map (a $\boxed{\text{"bijection"}}$)

$$T : F^n \longrightarrow V.$$

How is T defined?

① Define T on a basis $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_2}, \dots, \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_n} \right\}$

We set $T(\vec{e}_i) = v_i$.

② Say: "and extend T linearly."

$$T\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = T\left(a_1\vec{e}_1 + \dots + a_n\vec{e}_n\right) \overset{\text{additively}}{=} T(a_1\vec{e}_1) + \dots + T(a_n\vec{e}_n)$$

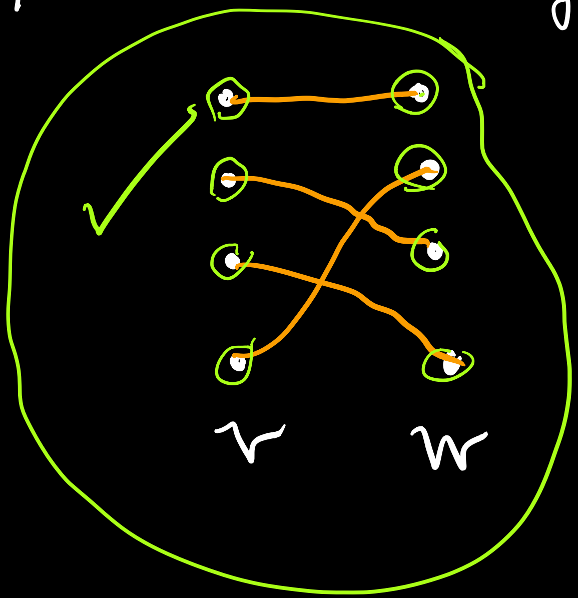
$$\overset{\text{pull out scalars}}{=} a_1 T(\vec{e}_1) + \dots + a_n T(\vec{e}_n)$$

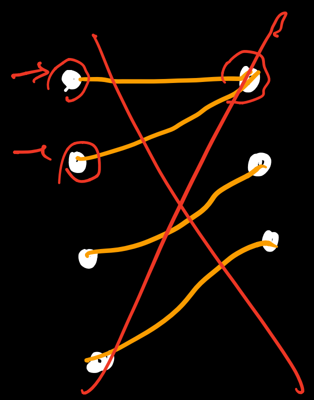
$$= a_1 v_1 + \dots + a_n v_n$$

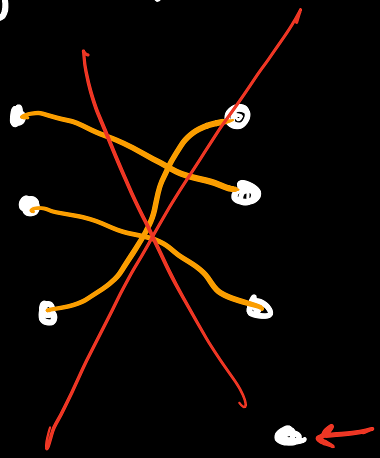
I haven't done anything

TAKEAWAY: T was completely determined by its behavior on $\{\vec{e}_1, \ldots, \vec{e}_n\}$.

A map $T: V \to W$ is bijective if every element of V is mapped to a unique element of W, and if every element of W is "hit by" some element of V.







A linear map from V to W is a map $T: V \to W$ that satisfies

vector spaces over the same field

- $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall\, v_1, v_2 \in V$

- $T(a \cdot v) = a \cdot T(v) \quad \forall\, a \in F,\ v \in V$

T is completely determined by its behavior on a basis.

Examples

used when specifying domain + range.

used when showing where an element goes.

$$T_a : \mathbb{R} \to \mathbb{R}, \quad a \in \mathbb{R} \text{ fixed}$$
$$v \mapsto a \cdot v$$

$$\left[T(c\cdot v + d\cdot w) = a\cdot(c\cdot v + dw) = a\cdot c\cdot v + a\cdot d\cdot w \right.$$
$$\left. = c\,T_a(v) + d\,T_a(w) \right]$$

$$T_a : \mathbb{R}^n \to \mathbb{R}^n$$
$$v \mapsto a\cdot v$$

T_1 is called the identity map

T_0 is called the zero map. It sends <u>everything</u> to 0.

Non-example

$T : \mathbb{R}^n \to \mathbb{R}^n$ over $F = \mathbb{R}$

$v \mapsto v + w$, $w \in \mathbb{R}^n$ is fixed, and non-zero.

$$T(v_1 + v_2) = v_1 + v_2 + w.$$

$$T(v_1) + T(v_2) = (v_1 + w) + (v_2 + w) = v_1 + v_2 + 2w$$

LINEAR MAPS ALWAYS SEND THE ADDITIVE IDENTITY TO THE ADDITIVE IDENTITY

Examples polynomials: $V = P(\mathbb{R})$

1. Pick $a \in \mathbb{R}$, define $T : P(\mathbb{R}) \longrightarrow \mathbb{R}$

$$p \longmapsto p(a)$$

"evaluation map at 'a'"

$$\left[T(p+q) = (p+q)(a) = p(a) + q(a) = T(p) + T(q) \right]$$

2. $T : P(\mathbb{R}) \longrightarrow P(\mathbb{R})$

$$p \longmapsto x \cdot p$$

3. $T : P(\mathbb{R}) \longrightarrow P(\mathbb{R})$

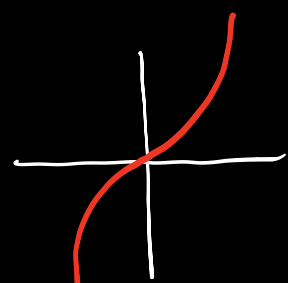
$$p \longmapsto \frac{dp}{dx}$$

NOTE

$$\left.\begin{array}{l} T : P(\mathbb{R}) \longrightarrow P(\mathbb{R}) \\ \quad p \longmapsto x^2 \cdot p \quad \text{is linear} \end{array}\right\}$$ a shift in basis

BUT $T : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto x^2 \cdot x$ is not linear



$1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad \ldots$

$T \downarrow$

$1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad \ldots$

This is a special case of a shift:

$$T : F^\infty \longrightarrow F^\infty$$

$$(x_1, x_2, \dots) \longmapsto (0, 0, x_1, x_2, \dots)$$

We denote by $\mathcal{L}(V, W)$ the set of linear maps from V to W. This has addition:

$$(T+S)(v) = T(v) + S(v)$$

and scalar multiplication:

$$(a \cdot T)(v) = a \cdot T(v).$$

Check for yourself: these operations make $\mathcal{L}(V, W)$ into a vector space.

Linear maps have additional structure:

<u>Composition</u>. $T \in \mathcal{L}(W, X)$ and $S \in \mathcal{L}(V, W)$ define $T \circ S \in \mathcal{L}(V, X)$ by

$$(T \circ S)(v) = T(S(v))$$

<u>NOTE</u>. Composition does not define a linear map

$$\mathcal{L}(V, W) \oplus \mathcal{L}(W, X) \longrightarrow \mathcal{L}(V, X).$$
$$(T, S) \longmapsto T \circ S.$$

<u>NOTE</u>. Composition is <u>not</u> commutative,

<u>ex</u>. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad$ $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x_1, x_2) \mapsto (x_2, -x_1) \qquad (x_1, x_2) \mapsto (x_1, -x_2)$$