Thm Rank-nullity theorem

$T \in \mathcal{L}(V, W)$, V and W are finite dimensional, then

$$\dim(V) = \dim \operatorname{null}(T) + \dim \operatorname{im}(T).$$

$\underline{Ex}$ $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}$

$(x,y) \longmapsto x$.

We computed: null $= \{(0,y) \mid y \in \mathbb{R}\}$

Rank-nullity thm: $\dim(\mathbb{R}^2) = \dim \text{null} + \dim \text{im}(T)$

$$2 = 1 + \dim \text{im}(T)$$

$$1 = \dim \text{im}(T).$$

$\underline{\text{Exercise}}$. $\mathcal{U} \subseteq V$ is a subspace and $\dim(\mathcal{U}) = \dim(V)$ then $\mathcal{U} = V$.

Checks out: staring at T, $\text{im}(T) = \mathbb{R}$.

$T: V \to W$ is surjective if

$$\dim(W) = \dim(V) - \dim \text{null}(T)$$

Ex $F^n \xrightarrow{T} F^n$

$x \longmapsto a \cdot x, \quad a \neq 0.$

$\mathrm{null}(T) = \{0\}.$

$\dim(F^n) = \dim \mathrm{null}(T) + \dim \mathrm{im}(T)$

$n = \dim \mathrm{im}(T).$

$\iff \mathrm{im}(T) = F^n.$

Ex $F^\infty \xrightarrow{T} F^\infty$

$(x_1, x_2, \ldots) \longmapsto (x_2, x_3, \ldots)$

$\text{null}(T) = \{(x_1, 0, 0, \ldots) \mid x_1 \in F\}$

$\dim \text{null}(T) = 1.$

F^∞ is infinite dim.

T is a surjection $\rightsquigarrow$ $\text{im}(T) = F^\infty$ is also infinite dimensional.

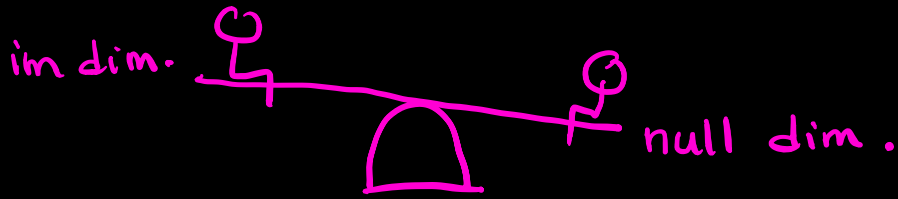
Prop.

(1) $\dim(V) > \dim(W) \Rightarrow \nexists$ a linear map $T: V \to W$ that is injective.

(2) $\dim(V) < \dim(W) \Rightarrow \nexists$ a linear map $T: V \to W$ that is surjective.

(3) $\dim(V) = \dim(W) \Rightarrow$ a linear map $T: V \to W$ is injective if and only if it is surjective.

My visualization of the Rank-nullity thm:



Pf of 3 : Let $T: V \to W$ be a linear map, and suppose $\dim(V) = \dim(W)$.

Rank-nullity thm :

$$(*) \qquad \dim(V) = \dim \operatorname{null}(T) + \dim \operatorname{im}(T).$$

($\Rightarrow$) Suppose T is injective.

Then $\operatorname{null}(T) = \{0\}$

$\rightsquigarrow$ $(*)$ becomes $\dim(V) = \dim \operatorname{im}(T) \subseteq W$.

$\rightsquigarrow$ Since $\dim(V) = \dim(W)$,

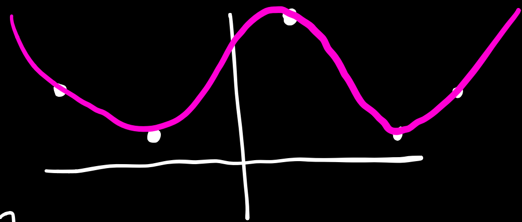
$$\dim \operatorname{im}(T) = \dim(W).$$

But $\operatorname{im}(T) \subseteq W$ is a subspace.

Exercise. $\Rightarrow$ $\operatorname{im}(T) = W$. Also ($\Leftarrow$) $\square$

Application: There always exists a degree $\leq d$ polynomial through $d+1$ distinct points. $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$

distinct x-values



Recall $\dim\left[P_d(\mathbb{R})\right] = d+1$.

Define $T : P_d(\mathbb{R}) \longrightarrow \mathbb{R}^{d+1}$

$$p \longmapsto (p(x_1), p(x_2), \ldots, p(x_{d+1}))$$

non-zero

Recall. A polynomial $p \in P_d(\mathbb{R})$ has at most d roots.

$\Rightarrow \operatorname{null}(T) = \{0\}$.

Rank-nullity: $\dim \operatorname{im}(T) = \dim P_d(\mathbb{R}) = d+1 = \dim \mathbb{R}^{d+1}$

$\Rightarrow \operatorname{im}(T) = \mathbb{R}^{d+1} \Rightarrow T$ is surjectivity.

$\Rightarrow \exists\ p \in P_d(\mathbb{R})$ with $T(p) = (y_1, y_2, \ldots, y_{d+1})$.

Application : Homogenous systems of linear equations

Suppose we have :

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n = 0$$

Want to find $x_1, x_2, \dots, x_n$ satisfying all k equations.

Rephrase : define a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$

by $T((x_1, \dots, x_n)) = (a_{11}x_1 + \dots + a_{1n}x_n, \dots\dots, a_{k1}x_1 + \dots + a_{kn}x_n)$

$\{\text{solutions}\} = \text{null}(T)$.

A solution exists if $\text{null}(T) \neq \{0\}$

k

Suppose $n > k$.

Then $\dim \mathrm{null}(T) = \dim(\mathbb{R}^n) - \overbrace{\dim(\mathrm{im}(T))}^{\leq}$

$$\geq n - k$$

$$> 0.$$

A solution <u>always</u> exists if $n > k$.

<u>NOTE</u>

(1) $\mathrm{null}(S \circ T) \supseteq \mathrm{null}(T)$

(2) $\mathrm{im}(S \circ T) \subseteq \mathrm{im}(S)$

<u>FUN</u>: How do we "see" null spaces and image spaces in terms of matrices and graphs?