Chapter 7 : Adjoints

Let A be an $m \times n$ matrix over $\mathbb{R}$ $\rightsquigarrow$ A^T

e.g. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$

A-dagger

Let A be an $m \times n$ matrix over $\mathbb{C}$ $\rightsquigarrow$ $A^\dagger$

$A^\dagger$ is the Hermitian conjugate

e.g. $\begin{bmatrix} 1 & 2i & 3 \\ 4 & 5 & 6 \end{bmatrix}^\dagger = \begin{bmatrix} 1 & 4 \\ -2i & 5 \\ 3 & 6 \end{bmatrix}$

Note: If A happens to have real entries, $A^\dagger = A^T$.

You might recall: $\underset{\text{real}}{(A\vec{x})\cdot\vec{y}} = \vec{x}\cdot(A^T\vec{y})$, $\vec{x} \in \mathbb{R}^n$

Prop. Let V, W be two finite-dim. inner product spaces. Let $\varphi : V \to W$ be a linear transformation.

$\exists!$ $\varphi^* : W \to V$ linear, satisfying

$$\langle \varphi(v), w \rangle = \langle v, \varphi^*(w) \rangle \quad \forall v \in V, w \in W.$$

We call φ^* the adjoint of φ.

Check: A $m\times n$ matrix $\rightsquigarrow$ $A : \mathbb{R}^n \to \mathbb{R}^m$

$A^T : \mathbb{R}^m \to \mathbb{R}^n$, characterized by

$$A\vec{x} \cdot \vec{y} = \vec{x} \cdot A^T\vec{y}$$

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^T\vec{y} \rangle.$$

<u>Ex</u> Axler 7.4: Let's fix $u, x \in V$

Define $T : V \to V$ by $Tv = \langle v, u \rangle x$.

What is T^*?

For $w \in V$

By defn.

$$\langle v, T^* w \rangle = \langle Tv, w \rangle$$

$$= \langle \langle v, u \rangle x, w \rangle$$

$$= \langle v, u \rangle \langle x, w \rangle$$

$= \langle v, \overline{\langle x, w\rangle} u \rangle.$

$\rightsquigarrow$ Define $T^* w = \overline{\langle x, w\rangle} u$.

Feels weird ...

However, we are told that

$$\langle v, T^* w\rangle = \langle v, \overline{\langle x, w\rangle} u\rangle \quad \forall_{?}\ v, w.$$

Proof of Prop.

Recall: Riesz Rep. Thm. says that if $\varphi : V \to \mathbb{F}$ is a linear map, then $\exists!\ u \in V$ s.t. $\varphi(v) = \langle v, u\rangle$.

In particular, define $\varphi_w : V \to \mathbb{F}$

$$v \mapsto \langle Tv, w\rangle_W$$

By RRT, $\exists!\ u \in V$ s.t. $\underbrace{\varphi_w(v) = \langle v, u \rangle}$.

$\rightsquigarrow \quad \langle Tv, w \rangle_W = \langle v, u \rangle_V$

Set $T^* w = u$.

This defines a map $T : V \to V$ satisfying

$$\underbrace{\langle Tv, w \rangle} = \underbrace{\langle v, T^* w \rangle} \quad \forall\ v, w.$$

Proof of linearity:

Additivity:
$$\begin{aligned}
\langle v, T^*(w_1 + w_2) \rangle &= \langle Tv, w_1 + w_2 \rangle \\
&= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle \\
&= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle
\end{aligned}$$

$$= \langle v, T^* w_1 + T^* w_2 \rangle.$$

Since this holds $\forall\, v \in V$, $T^*(w_1 + w_2) = T^* w_1 + T^* w_2$.

Scalar multiplicity: $\langle v, T^*(\lambda w) \rangle = \langle Tv, \lambda w \rangle$

$$= \bar{\lambda} \langle Tv, w \rangle$$

$$= \bar{\lambda} \langle v, T^* w \rangle$$

$$= \langle v, \lambda T^* w \rangle$$

$\rightsquigarrow$ $T^*(\lambda w) = \lambda T^*(w)$.

$\square$

Properties of adjoints

$\varphi, \psi : \mathcal{U} \to \mathcal{V}$ linear maps

$\gamma : \mathcal{V} \to \mathcal{W}$

(1) $(\varphi + \psi)^* = \varphi^* + \psi^*$ $\qquad (A+B)^T = A^T + B^T$

(2) $(\lambda\varphi)^* = \bar{\lambda} \cdot \varphi^* \quad \forall \lambda \in \mathbb{F}$ $\qquad (cA)^T = c \cdot A^T$

$\boxed{(3)}$ $(\varphi^*)^* = \varphi$ $\qquad (A^T)^T = A$

$\boxed{(4)}$ $(\gamma \circ \varphi)^* = \varphi^* \circ \gamma^*$ $\qquad (A \circ B)^T = B^T \circ A^T.$

Proof of 3

$$\langle \varphi^* v, w \rangle = \overline{\langle w, \varphi^* v \rangle}$$

$$= \overline{\langle \varphi w, v \rangle}$$

$$= \langle v, \underbrace{\varphi w}_{(\varphi^*)^*(w)} \rangle$$

Proof of 4

$$\langle (\gamma \circ \varphi)(v), w \rangle = \langle \gamma(\varphi(v)), w \rangle$$

$$= \langle \varphi(v), \gamma^* w \rangle$$

$$= \langle v, \varphi^* \circ \gamma^* (w) \rangle$$