Objectives for today

1. Prove properties of V using the 8 Fundamental Properties

2. Define and give examples of subspaces

Let V be a vector space over a field F, w/ $+, \cdot$

By definition:

1. $\forall\ x, y, z \in V \quad (x+y)+z = x+(y+z)$
2. $\forall\ x, y \in V \quad x+y = y+x$
3. $\exists\ \vec{0}$ s.t. $\forall\ x \in V \quad x+\vec{0} = x$
4. $\forall\ x \in V\ \exists\ (-x) \in V$ s.t. $x+(-x) = \vec{0}$
5. $\forall\ c \in F$ and $x, y \in V \quad c\cdot(x+ +y) = c\cdot x + c\cdot y$
6. $\forall\ c, d \in F$ and $\forall\ x \in V \quad (c+d)\cdot x = c\cdot d + d\cdot x$
7. $\forall\ c, d \in F$ and $\forall\ x \in V \quad c\cdot(d\cdot x) = (cd)\cdot x$
8. $\forall\ x \in V, \quad 1\cdot x = x.$

Recall Efficiency is important!

What other properties could be desirable?

① Uniqueness

Is $\vec{0}$ unique? Is there another $\vec{0}' \in V$ s.t. $\vec{0}' + x = x$.

$\vec{0}$ is characterized by the property that $\vec{0} + x = x$.

Is $(-x)$ unique, given $x \in V$?

② Another: How does 0 behave?

How does $\vec{0}$ behave?

Ex. $0 \cdot x = \vec{0}$?

$c \cdot \vec{0} = \vec{0}$?

<u>Proposition</u> Let V be a vector space over a field F. Then

$\boxed{\text{(i)}}$ The additive identity $\vec{0}$ is unique.

HW

(ii) $\forall\ v \in V$ the additive inverse $-v$ is unique.

$\boxed{\text{(iii)}}$ For each $v \in V$ $\quad 0 \cdot v = \vec{0}$

(iv) For each $c \in F$ $\quad c \cdot \vec{0} = \vec{0}$.

ex.

Proof of (i) Proof by Contradiction.

Assume, for contradiction that $\vec{0}$ and $\vec{0}'$ are different additive identities.

Let $\vec{0}$ and $\vec{0}'$ be two additive identies in V.

Then $\vec{0}' = \vec{0}' + \vec{0}$ by definition of $\vec{0}$ as an additive identity

$= \vec{0} + \vec{0}'$ by commutativity

$= \vec{0}$ by definition of $\vec{0}'$ as an additive identity.

Thus, $\vec{0}' = \vec{0}$, and we reach a contradiction. ~~and the additive identity is unique.~~ $\square$

Proof of (iii)

Because 0 is the additive identity of F,

$$0 \cdot x = (0+0) \cdot x \quad \forall \quad x \in V.$$

trick: exploit special properties of $0 \in F$.

By distributivity,

$$(0+0) \cdot x = 0 \cdot x + 0 \cdot x.$$

Thus, $\quad 0 \cdot x = 0 \cdot x + 0 \cdot x \qquad (\star)$

Let $-0 \cdot x$ be the additive inverse of $0 \cdot x$ in V.

Adding $-0 \cdot x$ to both sides of $(\star)$,

$$0 \cdot x + (-0 \cdot x) = 0 \cdot x + 0 \cdot x + (-0 \cdot x).$$

Simplifying using the definition of additive inverse,

$$\vec{0} = 0 \cdot x + \vec{0}$$

$$= 0 \cdot x \quad \text{by definition of } \vec{0}. \quad \square$$


?
End.

__Prove or disprove__ : $\forall\ x, y \in F\ \ \exists\ z \in F$ such that $x + z + z = y$.

one defn : $y - x := y + (-x)$, where $-x$ is the additive inverse of x.

~~Pf~~. Define $z = \boxed{\frac{1}{2}}(y - x)$. Then

$$x + z + z = x + \tfrac{1}{2}(y-x) + \tfrac{1}{2}(y-x)$$
$$= x + y - x \quad \text{--- distributivity}$$
$$= x + y + (-x)$$
$$= x + (-x) + y$$
$$= 0 + y$$
$$= y + 0$$
$$= y.$$

BAD

WHAT IS TERRIBLE??

1. Lots of unjustified equalities
2. Subtraction hasn't been defined.
3. $\frac{1}{2}$ may not have meaning.

Pf. We will show that the statement is false by finding a counterexample.

Let $F = \mathbb{Z}_2 = \{0, 1\}, +, \cdot$ modular arithmetic

Let $x = 0$ and $y = 1$.

Then $x + z + z = 1 \iff$ if and only if

$z + z = 1$

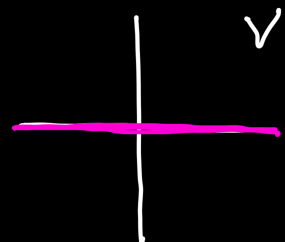
which is impossible because $0+0=0$ and $1+1=0$.

$\square$

SUBSPACES

Definition. A subspace $\mathcal{U}$ of a vector space V is a subset of V that is a vector space with operations $+, \cdot$ induced by $+, \cdot$ on V.

Ex 1. $\mathcal{U} = \{(x,0) \mid x \in \mathbb{R}\}$, $V = \mathbb{R}^2$.



Ex 2. $\mathcal{U} = \{(x,x,x) \mid x \in \mathbb{R}\}$, $V = \mathbb{R}^3$

Ex 3. $\mathcal{U} = \{\vec{0}\}$, $V = \mathbb{R}^n$.

Proposition $\mathcal{U} \subseteq V$ is a subspace of V

the set $\mathcal{U}$ is contained in the set V

if and only if the following three conditions hold:

(i) $\vec{0} \in \mathcal{U}$ (the additive identity of V is in $\mathcal{U}$)

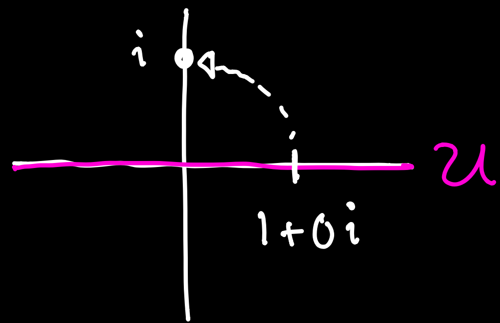
(ii) $\forall\, u, v \in \mathcal{U}$ $u + v \in \mathcal{U}$ ($\mathcal{U}$ is closed under addition)

(ii) $\forall\, u \in \mathcal{U}$ and $c \in \mathbb{F}$

$c \cdot u \in \mathcal{U}$. ($\mathcal{U}$ is closed under scalar multiplication)

<u>non-example</u>: $\mathcal{U} = \{ x + 0i \} \subseteq \mathbb{C}$, viewed as a vector space over $\mathbb{C}$.

$\mathcal{U}$ is <u>not</u> a subspace.



$$i \cdot (1 + 0i) = i + 0 i \cdot i = i \notin \mathcal{U}.$$

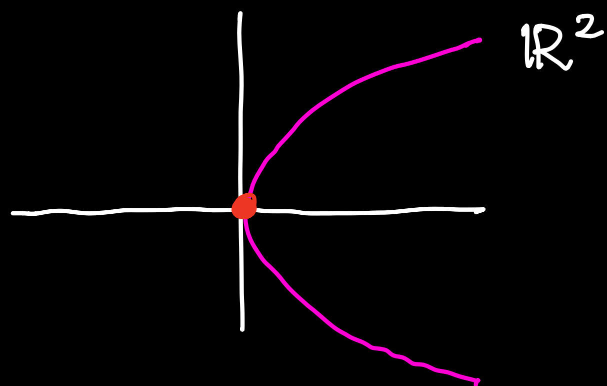
So $\mathcal{U}$ is not closed under scalar multiplication.

<u>Ex</u>. $U = \{(x,y,z) \in \mathbb{R}^3 \mid 5x+2y+z = 0\}$

is a subspace of $\mathbb{R}^3 = V$.

<u>Non-ex</u>. $U = \{(x,y) \in \mathbb{R}^2 \mid \underbrace{x - y^2 = 0}_{x=y^2}\}$ is

not a subspace of $\mathbb{R}^2$.



We had 3 properties to check.

Does closure under scalar multiplication hold?

(x,y) must satisfy $x-y^2=0$, for (x,y) to be in $\mathcal{U}$.

Let $c \in \mathbb{R}$. Is $c\cdot(x,y)=(cx,cy)$ in $\mathcal{U}$?

In other words, is $(cx)-(cy)^2=0$?

$\iff cx-c^2y^2=0$

$\iff c(x-cy^2)=0$.

Assume $c\neq 0$.

$\iff x-cy^2=0$.

Take $c=2$. If $x-cy^2=0$, then $x-2y^2=0$.

But by assumption, $x - y^2 = 0$, because $(x,y) \in U$.

Take $(x,y) = (1,1)$. Then $(x,y) \in U$ but $(2x,2y) = 2,2$ does not satisfy

$$x - y^2 = 0.$$