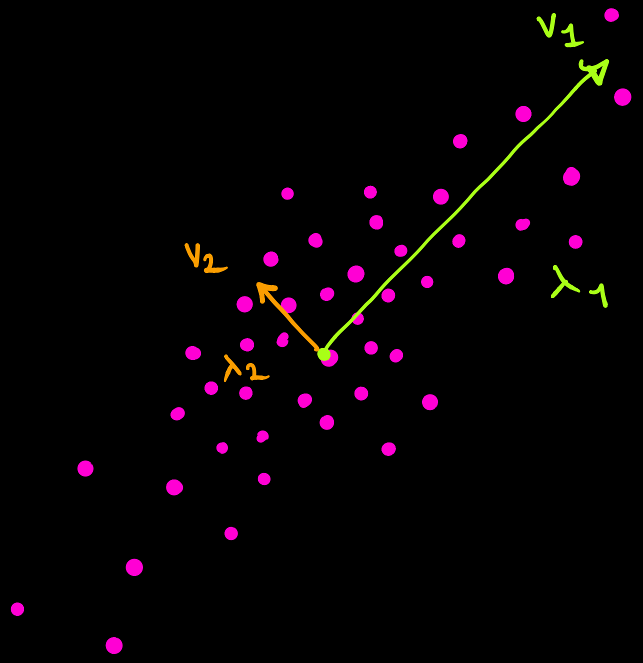
(2) If a matrix over a field F is diagonalizable then all eigenvalues of the matrix lie in F.

e.g. All eigenvalues of covariance matrix lie in $\mathbb{R}$.

(actually, they all lie in $\underbrace{\mathbb{R}_{\geq 0}}_{\{x \in \mathbb{R} \mid x \geq 0\}}$!)

Each eigenvalue gives a quantitative measure of how much the data stretches in various directions.

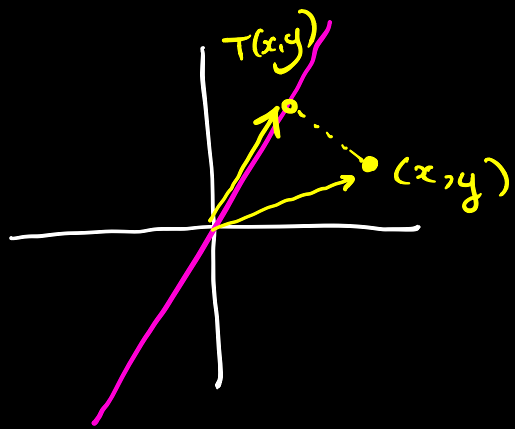


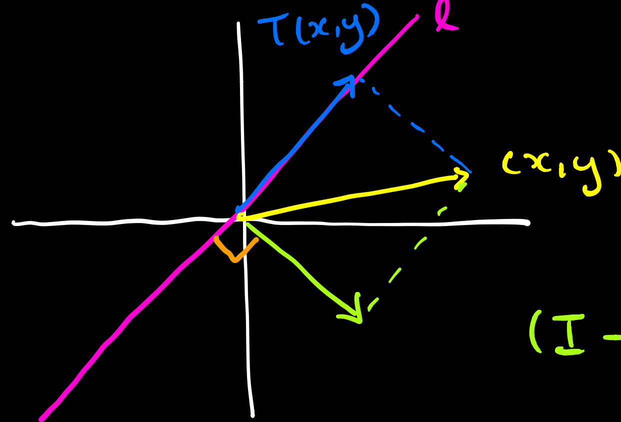
$\lambda_1, \lambda_2 \in \mathbb{R}$!

Ch 6. Recall from HW 3 an idempotent map is a $T \in \mathcal{L}(V)$ such that $T^2 = T$.

Ex. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is projection onto the x-axis.
$(x,y) \mapsto (x,0)$

Ex. Any projection map





$(I-T)(x,y) \in \ker(T)$

Characterized by:

(1) $\ker(T) \supseteq \operatorname{im}(I-T)$ generalizes immediately to all idempotent operators.

Restatement

(2) Right angle.

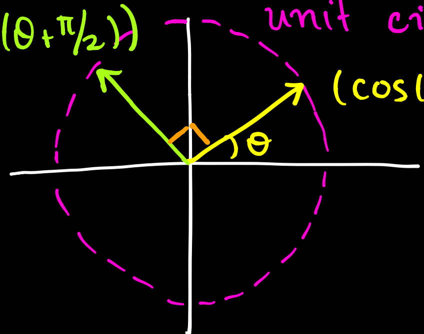
How do we generalize? What is a "right angle" in $P(\mathbb{R})$? Does this even make sense to ask?

Alternative formulation of angles in $\mathbb{R}^2$: via the <u>dot product</u>: a map $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

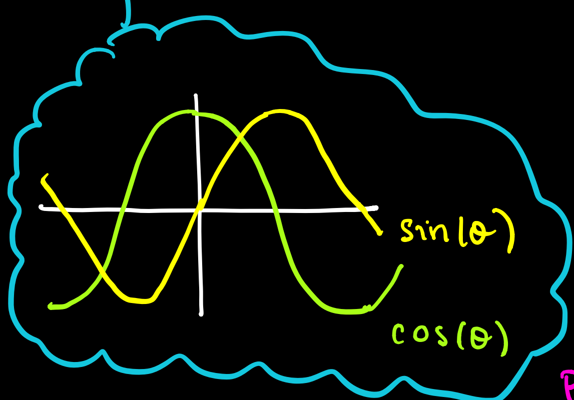
$$(x, y), (w, z) \longmapsto xw + yz.$$

$$\vec{x}_2 = \left(\cos(\theta + \pi/2), \sin(\theta + \pi/2)\right) = \left(-\sin(\theta), \cos(\theta)\right)$$

unit circle



$$(\cos(\theta), \sin(\theta)) = \vec{x}_1$$



$\vec{x}_1$ and $\vec{x}_2$ are perpendicular

and $\vec{x}_1 \cdot \vec{x}_2 = \cos(\theta)(-\sin(\theta)) + \cos(\theta)\sin(\theta) = 0.$

TAKEAWAY: $\vec{x}$ and $\vec{y}$ are perpendicular when $\vec{x} \cdot \vec{y} = 0$.

This looks like something we can generalize.

(1) How might we define a dot product on $\mathbb{R}^n$?

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \ldots + x_n y_n .$$

(2) How might we define a dot product on $\mathbb{C}^n$?

$$(w_1, \ldots, w_n) \cdot (z_1, \ldots, z_n) = w_1 \bar{z}_1 + \ldots + w_n \bar{z}_n .$$

$z_1 = a + bi, \quad \bar{z}_1 = a - bi.$

(3) How might we define a dot product on $\mathcal{F}([-1,1])$?

(4) How might we define a dot product on F^n, where F is a finite field?

you can't!

We call "dot products" inner products.

Def. An inner product on a vector space V over $\overbrace{\mathbb{R} \text{ or } \mathbb{C}}^{F}$ is a function $V \times V \to F$ that has the following properties
$$(v, w) \mapsto \langle v, w \rangle$$

can't work over a finite field.

(1) positivity: $\langle v, v \rangle \in \mathbb{R}_{\geq 0} \quad \forall \ v \in V$

captures size

(2) definiteness: $\langle v, v \rangle = 0 \iff v = 0$.

a non-zero vector can't be perpendicular to itself.

(3) additivity: $\langle v + w, z \rangle = \langle v, z \rangle + \langle w, z \rangle$

$\langle v, w + z \rangle = \langle v, w \rangle + \langle v, z \rangle$

respects v.s. operations

(4) homogeneity: $\lambda \in F$ $\quad \langle \lambda v, w \rangle = \lambda \langle v, w \rangle$

$\langle v, \lambda w \rangle = \overline{\lambda} \langle v, w \rangle$

angle doesn't depend on order.

(5) conjugate symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

written in decreasing importance (from my point of view)

NOTE $\langle v, 0\rangle = \langle v, 0+0\rangle = \langle v, 0\rangle + \langle v, 0\rangle$

So $\langle v, 0\rangle = 0$.

Similarly, $\langle 0, v\rangle = 0$.

Why the conjugate condition?

If $F = \mathbb{R}$ then $x = \overline{x}$. So (4) and (5) become

$$\langle x, \lambda w\rangle = \lambda\langle x, w\rangle \quad \text{and} \quad \langle x, w\rangle = \langle w, x\rangle.$$

If $F = \mathbb{C}$, say $V = \mathbb{C}$ we could try to define

$$\langle w, z\rangle = w \cdot z.$$

However, writing $w = a + bi$,

$$\langle w, w\rangle = (a+bi)(a+bi) = \underbrace{a^2 - b^2 + 2abi}_{\text{is not in } \mathbb{R}_{\geq 0}}.$$

So instead, we define $\langle w, z \rangle = w\bar{z}$.

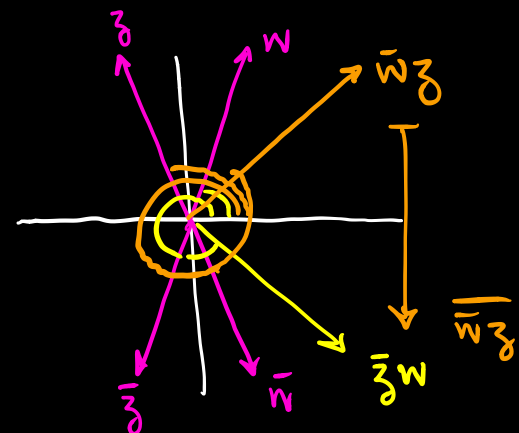
Then $\langle w, w \rangle = (a+bi)(a-bi) = a^2 + b^2$.

(5) $\langle w, z \rangle = w\bar{z}$

$$= \bar{z}w$$

$$= \overline{z\bar{w}}$$

$$= \overline{\langle z, w \rangle}$$



(4) $\langle w, \lambda z \rangle = w\overline{\lambda z}$

$$= \bar{\lambda} w \bar{z}$$

$$= \bar{\lambda} \langle w, z \rangle.$$

<u>Def</u> A vector space equipped with an inner product an <u>inner product space</u>.

Back to functions:

We can think of a vector space V of $\dim(V) = n$ as the set of functions $\{1, 2, \ldots, n\} \to F$.

Choose a basis of V $\{v_1, \ldots, v_n\}$ and think of a vector $a_1 v_1 + \ldots + a_n v_n$ as the function

$$i \longmapsto a_i.$$

e.g. $\mathbb{R}^n$, standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$.

The vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ as the function $i \longmapsto a_i$.

$$\underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\varphi_a} \cdot \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\varphi_b} = a_1 b_1 + \ldots + a_n b_n = \varphi_a(1)\varphi_b(1) + \ldots + \varphi_a(n)\varphi_b(n)$$

$$= \sum_{i=1}^{n} \varphi_a(i)\varphi_b(i).$$

Can I extend this observation to define an inner product on $\mathcal{F}([-1,1])$?

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\cdot g(x)\,dx.$$

$P(\mathbb{R})$? $\quad \langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\cdot g(x)\, e^{-x^2}\,dx.$

some weight here to make my integral defined.

Other inner products on $\mathbb{R}^n$, $\mathbb{C}^n$

e.g. $\langle (x,y), (w,z)\rangle = a\,x w + b\,y z$

$$a, b \in \mathbb{R}_{>0}$$