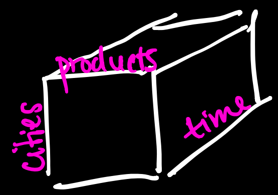
Multi-dimensional data

[data cube, OLAP cube ⟵  ⟵ 3-tensor

Tensor - array of any dimension.

Matrix - 2-tensor

vector - 1-tensor

scalar - 0-tensor

Q. Can we use mathematical language (in particular, the language of linear algebra) to talk about tensors?

YES: a few different ways to proceed

$\boxed{1}$ If I have $v, w \in \mathbb{R}^n$, then $vw^T = A$ a $n \times n$ matrix.

e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

ith slot $\rightarrow \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

In general, $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \sum_{i,j} a_{ij} \underbrace{e_i e_j^T}$

important! let's introduce terminology

e.g. $3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

$3 \cdot 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = 6 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix}$

$+ \Big\} \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix}$

We write $A = \sum_{i,j} a_{ij} e_i \otimes e_j$ [notation, means the tensor product same as above]

Q: What should $v \otimes w$ mean?

Guess 1: $v \otimes w = vw^T$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Guess 2: $v = e_1 + 2e_2$

$w = 3e_1 + 4e_2$

$$\begin{aligned} v \otimes w &= (e_1 + 2e_2) \otimes (3e_1 + 4e_2) \\ &= e_1 \otimes (3e_1 + 4e_2) + 2e_2 \otimes (3e_1 + 4e_2) \\ &= \boxed{3e_1 \otimes e_1 + 4e_1 \otimes e_2 + 6e_2 \otimes e_1 + 8e_2 \otimes e_2} \end{aligned}$$

$$= \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Our guesses coincide!

How do we formalize this?

1. (Free product) $F(V \times V) = \left\{ \sum_{i=1}^{l_i} (v_{1_i}, v_{2_i}) \right\}$

2. (Quotient) $F(V \times V) \big/ \sim$

equivalence relation

where $(v, w) \sim (v, w)$

$(v, w) \sim (u, x) \iff (u, x) \sim (v, w)$

$(v,w) \sim (u,x)$ AND $(u,x) \sim (y,z) \Rightarrow (v,w) \sim (y,z)$

and

$(v_1 + v_2, w) \sim (v_1, w) + (v_2, w)$

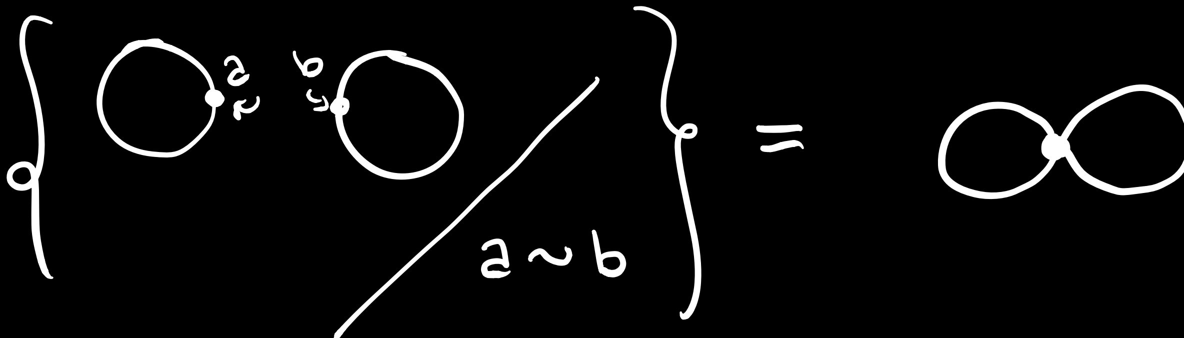
$(cv, w) \sim (v, cw)$

$$V \otimes V := F(V \times V) / \sim$$

" $V \otimes V$ is all elements in $F(V \times V)$, where 'by hand' we decree that $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$

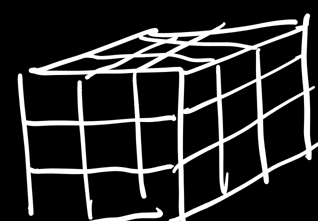
and $(cv, w) = (v, cw)$."

A geometric picture:



Zipshot You can think of a 3-tensor as

$$\sum_{ijk} a_{ijk}\, e_i \otimes e_j \otimes e_k$$



<u>or</u> you can think of a 3-tensor as an element of $V \otimes V \otimes V =: V^{\otimes 3}$

"By abstract reasons tensor products exist and are well-defined".

Universal Properties

Penrose Graphical Notation

tensor networks

Exterior algebras and determinants.

Tensor products are quite big... let's look at simpler pieces.

1) "Diagonal tensor": things of the form

I made this up

$$\sum_{i=1}^{n} a_{ii} e_i \otimes e_i .$$

2) Quotient by diagonal tensors:

$$V \wedge V = V \otimes V \Big/ v \otimes v \sim 0 .$$

wedge

called the exterior algebra.

Looks a bit strange, but it has the following important property:

$$\boxed{v \wedge w = -w \wedge v.}$$

"anti-commutative, aka, switching the order switches the sign."

What does this have to do with determinants?

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$A(e_1) \wedge A(e_2) = (ae_1 + be_2) \wedge (ce_1 + de_2)$$

$$= ae_1 \wedge (ce_1 + de_2) + be_2 \wedge (ce_1 + de_2)$$

$$= ae_1 \wedge ce_1 + ae_1 \wedge de_2 + be_2 \wedge ce_1 + be_2 \wedge de_2$$

$$= \underbrace{ac\, e_1 \wedge e_1}_{=0} + ad\, e_1 \wedge e_2 + bc\, \underbrace{e_2 \wedge e_1}_{-e_1 \wedge e_2} + bd\, \underbrace{e_2 \wedge e_2}_{=0}$$

$$= (ad - bc)\, e_1 \wedge e_2$$

$$= \det(A)\, e_1 \wedge e_2 .$$

Define $\det(T)$, $T \in \mathcal{L}(V)$, to be the scalar satisfying $T(v_1) \wedge \dots \wedge T(v_n) = \det(T)\, v_1 \wedge \dots \wedge v_n$.

Think calc: $\iint dx\, dy \rightsquigarrow \int \underbrace{dx \wedge dy}$