RECALL: Last time we saw that self-adjoint operators admit $\underbrace{\text{orthonormal eigenvector bases}}_{\text{O.E.B.}}$ (proved for $F = \mathbb{C}$)

We asked: What are necessary and sufficient conditions for an operator T to have an O.E.B.?

Introduced: normality: (1) $TT^* = T^*T$

(2) $\|Tv\| = \|T^*v\| \quad \forall \, v \in V$.

Prop 1. Defn (1) and Defn (2) are equivalent.

Prop 2. Eigenvectors corresponding to distinct eigenvalues of normal T are orthogonal.

Pf. $\alpha \neq \beta$ $\quad Tv = \alpha v$ and $Tw = \beta w$

$$(\alpha - \beta)\langle v, w\rangle = \langle \alpha v, w\rangle - \langle v, \bar{\beta} w\rangle$$

$$= \langle Tv, w\rangle - \langle v, \bar{\beta} w\rangle$$

$$= \langle v, T^* w\rangle - \langle v, \bar{\beta} w\rangle$$

$$= \langle v, T^* w - \bar{\beta} w\rangle$$

~~If $T^* w = \bar{\beta} w$, then $\langle v, T^* w - \bar{\beta} w\rangle = 0$~~
~~and we are done. $[(\alpha - \beta)\langle v, w\rangle = 0]$.~~

By the Lemma, $T^* w - \bar{\beta} w = 0$. The result follows.

$\#$

Lemma. If T is normal and $Tw = \beta w$, then $T^* w = \bar{\beta} w$.

"Pf."

(1 Attempt) I know $TT^* = T^* T$.

$$T(T^*(w)) = T^*(T(w)) = T^*(\beta w) = \beta T^*(w).$$

(This shows that $T^* w$ is an eigenvector of T with eigenvalue β)

(2 Attempt) I know $\|Tw\| = \|T^* w\|$.

(3 Attempt) Look at Axler's proof

Clever observation: $(T - \lambda I)^* = T^* - \bar{\lambda} I$.

[Adjoints are additive and $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle = \langle v, \bar{\lambda} w \rangle$]

Another clever observation: $(T - \lambda I)$ is normal.

$$
\begin{aligned}
(T - \lambda I)(T - \lambda I)^* &= (T - \lambda I)(T^* - \bar{\lambda} I) \\
&= TT^* - \lambda I T^* - \bar{\lambda} T I + \lambda \bar{\lambda} I^2 \\
&= (T^* - \bar{\lambda} I)(T - \lambda I) \\
&= (T - \lambda I)^*(T - \lambda I)
\end{aligned}
$$

So
$$
\begin{aligned}
0 = \| (T - \beta I) w \| &= \| (T - \beta I)^* w \| \\
&= \| (T^* - \bar{\beta} I) w \|.
\end{aligned}
$$

$\rightsquigarrow (T^* - \bar{\beta} I)w = 0$, or $T^* w = \bar{\beta} w$. $\square$

In English: T and T^* have the same eigenvectors

If w is an eigenvector of T with eigenvalue β

them" " T^* " " $\bar{\beta}$.

SPECTRAL THEOREM $\boxed{(\text{over } \mathbb{C})}$

$T \in \mathcal{L}(V)$ is a normal operator if and only if it admits an orthonormal basis of eigenvectors.

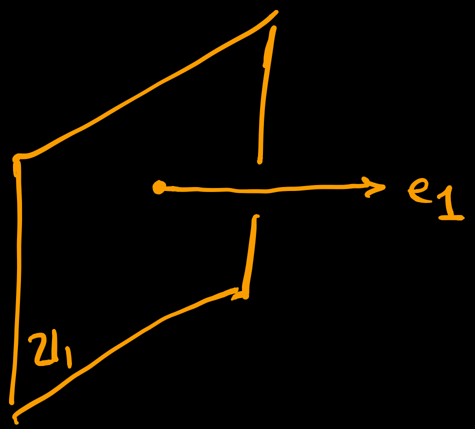
What is our method of proof?

Recall: if T is self-adjoint, T admits an O.E.B.

How did we prove this?

We inducted on dimension.

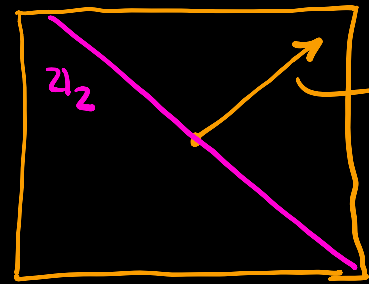
Geometrically: we knew that T had some eigenvectors



$\mathcal{U}_1 = \text{span}(e_1)^{\perp}$

$\boxed{\text{Self-adjointness} \Rightarrow T|_{\mathcal{U}_1} \subseteq \mathcal{U}_1.}$

So restrict T to $\mathcal{U}_1$:



e_2 eigenvector of $T|_{\mathcal{U}_1}$.

$\mathcal{U}_2 = \text{span}(e_2)^\perp$.

And repeat....

At the end of the day... $\{e_1, e_2, \dots, e_n\}$ an orthogonal basis of eigenvectors.

We can recycle this proof! We just need to show that T preserves $\mathcal{U}_i$, and that T^* preserves $\mathcal{U}_i$.

Let $u \in \mathcal{U}$.

e_1 is an eigenvector with eigenvalue β.

By Lemma, e_1 is an eigenvector of T^* with eigenvalue $\bar{\beta}$.

def. of adjointness. (Aside: when we proved this for self-adjoint operators, this string of equalities was:

$$\langle Tu, e_1\rangle = \langle u, T^* e_1\rangle = \langle u, Te_1\rangle = \langle u, \beta e_1\rangle.$$

Lemma used

$$\begin{aligned}\langle Tu, e_1\rangle &= \langle u, T^* e_1\rangle \\ &= \langle u, \bar{\beta} e_1\rangle \\ &= \beta \langle u, e_1\rangle \\ &= 0.\end{aligned}$$

$Tu \in \operatorname{span}(e_1)^\perp \implies Tu \in \mathcal{U}$.

Sketch

To prove that T has an O.E.B. $\Rightarrow$ T is normal

let A be the matrix of T w.r.t. this basis.

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

The matrix of T^* w.r.t. the same basis is $A^* = \begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix}$.

Diagonal matrices commute, so $TT^* = T^*T$.

$\boxed{\text{Sketch}}$

Textbook proof:

1) Uses induction on dimension technique to show in 5.27 that all operators have an upper triangular matrix in some basis.

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

2) Schur's Lemma: You can choose this basis to be orthonormal.

3) Spectral Thm pf: the "upper triangular matrix" of a normal operator is diagonal.